How to apply Tail Value at Risk over multiple time steps avoiding accumulation of conservatism and extra parameters

Berend Roorda* J.M. Schumacher†

Extended abstract for the Bachelier Finance Society World Congress
July 2008, London

Abstract

We compare two different definitions of Tail-Value-at-Risk in multiperiod models: the well-known backward recursive dynamically consistent definition (DTVaR), and the sequentially consistent version (STVaR), satisfying a weaker form of time consistency that only requires that acceptability levels do not exceed essential extremes later on. We extend the proposal for STVaR in Roorda and Schumacher (2007) to a more general setting, and indicate why STVaR is to be preferred if low levels (conditioning on tails with low probability) plays a role, as e.g. in a regulatory context. We show how the backward recursion in DTVaR can be restored for STVaR, to some extent, and indicate how this can be exploited in evaluation by Monte-Carlo simulation.

We compare two different definitions of Tail-Value-at-Risk in multiperiod models: the well-known backward recursive dynamically consistent definition (DTVaR), and the sequentially consistent version (STVaR), satisfying a weaker form of time consistency that only requires that acceptability levels do not exceed essential extremes later on. We indicate why STVaR is to be preferred if low levels (conditioning on tails with low probability) plays a role, as e.g. in a regulatory context. It is shown how the backward recursion in DTVaR can be restored for STVaR, to some extent, and we indicate how this can be exploited in evaluation by Monte-Carlo simulation.

Keywords: convex risk measures; acceptability measures; weak time consistency; Tail Value at Risk.

1 Introduction

We take our starting point in the well-known single-period risk measure *Tail Value at Risk* (TVaR), also known as Expected Shortfall, Conditional Value at Risk, or Average Value at Risk (see e.g.

*B. Roorda, FELab and School of Management and Governance, University of Twente, P.O. Box 217, 7500 AE, Enschede, the Netherlands. Phone: +3153-4894383. E-mail: b.roorda@utwente.nl. Responsible for this abstract.

†J.M. Schumacher, CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, the Netherlands.

Föllmer and Schied (2004), Pflug (2007), and the references therein). We prefer to define TVaR as an acceptability measure, with acceptability being the negative of riskiness. For a given probability space $(\Omega, \mathcal{F}, P)$, let $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ represent the set of all financial positions under consideration. Then TVaR at level $c \in [0,1]$, under reference measure P, is defined by

$$\text{TVaR}_c(X) := \inf\{E_P[LX] | L \in L^\infty, 0 \leq L \leq c^{-1}, E_P[L] = 1\} \tag{1.1}$$

or, in dual form,

$$\text{TVaR}_c(X) = \inf\{E_Q[X] | \frac{dQ}{dP} \leq c^{-1}\} \tag{1.2}$$

where $\frac{dQ}{dP}$ denotes the density of Q with respect to P. This amounts to considering the average of X over the worst c probability mass, if $(\Omega, \mathcal{F}, P)$ is non-atomic.

It is important to notice that the interpretation of the outcome of $\text{TVaR}_c(X)$ heavily depends on the level c. For $c = 1$, and P a martingale measure, it would correspond to an arbitrage-free price for X. For the other extreme, $c = 0$, it amounts to the worst outcome of X, hence the cash amount needed for full protection against any loss. In between the interpretation gradually changes from risk-adjusted prices, for c not far away from 1, to required risk capital for providing protection against losses up to a certain confidence bound, typically with $c = 0.05$, or even $c = 0.0001$ for annual Economic Capital models at AAA-rated banks. Rather than prices, the corresponding amounts for c close to zero represent cash amounts that are most probably never to be used for X; they are by no means a sound basis for exchanging X for cash. This difference between high and low levels is important in the sequel.

An obvious way to extend TVaR to a multiperiod model is by applying TVaR backward recursively over each step. We call this Dynamic TVaR (DTVaR). So we consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, P)$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$. $L_t^\infty = L^\infty(\Omega, \mathcal{F}_t, P)$ is the set of $\mathcal{F}_t$-measurable random variables. A multiperiod acceptability measure is a family $\Phi = (\phi_t)_{t=0,\ldots,T}$ with $\phi_t : L^\infty \to L_t^\infty$. We write ϕ for ϕ_0. In the sequel, all inequalities and equalities applied to random variables are understood to hold P-almost surely.

The initial measure $\text{DTVaR}_{c,0}$, or DTVaR_c for short, can be expressed as

$$\text{DTVaR}_c(X) := \inf\{E_P[X\Pi_{s=1}^T L_s] \,|\, L_s \in L_s^\infty, 0 \leq L_s \leq c^{-1}, E_P[L_s|\mathcal{F}_{s-1}] = 1 \text{ for } s = 1, \ldots, T\}. \tag{1.3}$$

At later time, $\text{DTVaR}_{c,t}$ allows for a similar expression, with expectation now conditional on $\mathcal{F}_t$, and s starting in $t+1$. Dual formulations are obtained, analogous to (1.2), by letting L_s correspond to $\frac{E_P[dQ/dP|\mathcal{F}_s]}{E_P[dQ/dP|\mathcal{F}_{s-1}]}$, cf. Artzner et al (2007).

2 What is the problem with DTVaR?

There is no problem as long as the outcome of $\phi_t(X)$ can be interpreted as the value of X at time t. Then one could exchange X for $\phi_t(X)$ at time t, and, ignoring that this cannot be reversed in general, it makes sense to impose the rule that the acceptability of X must be the same as the acceptability of its value at time t,

$$\phi(X) = \phi(\phi_t(X)).$$

This is the axiom of strong time consistency (see Frittelli and Rosazza Gianin (2004), Roorda and Schumacher (2007), Föllmer and Penner (2006), Artzner et al (2007), which essentially reduces multiperiod acceptability to backward recursive static acceptability. It is clear that DTVaR satisfies this axiom.

However, we already indicated that for low values of c, the outcome of $\mathrm{DTVaR}_c(X)$ cannot be treated as values of X anymore, and then strong time consistency becomes less appropriate. In fact, the axiom would lead to an absurd accumulation of conservatism over the entire period. As an illustration, one could consider $\mathrm{DTVaR}_c(X)$ for $c = 0.01$, $T = 10$, and X lognormally distributed. This is the average value of X assuming a sequence of ten extreme downturns at a row, which amounts taking a conditional expectation $E_P[X|B]$ with $P(B) = 10^{-20}$. We refer to Roorda and Schumacher (2007) for another small example illustrating this point.

3 What then is an appropriate notion of time consistency?

It is clear that strong time consistency is incompatible with the combination of low levels and many time steps. Therefore, we consider 'weaker' forms of time consistency, that, on the one hand, still guarantee some desirable properties of the measure, but, on the other hand, are more flexible than the certainty equivalent principle.

We consider the following notions of weak time consistency.

Definition 3.1 $\Phi = (\phi_t)_{t=0,\dots,T}$ is said to be *conditionally consistent* if it satisfies the following condition for all $t = 0, \dots, T-1$ and all $X \in L^\infty$:

$$\phi_{t+1}(X) \geq 0 \Leftrightarrow \text{ for all } F \in \mathcal{F}_{t+1}, \phi_t(X \mathbb{1}_F) \geq 0. \tag{3.1}$$

Definition 3.2 $\Phi = (\phi_t)_{t=0,\dots,T}$ is said to be *sequentially consistent* if it satisfies the following two conditions for $t = 0, \dots, T-1$, and all $X \in L^\infty$:

1. $\phi_{t+1}(X) \geq 0 \Rightarrow \phi_t(X) \geq 0$ ('acceptance consistency')
2. $\phi_{t+1}(X) \leq 0 \Rightarrow \phi_t(X) \leq 0$ ('rejection consistency')

These notions have been analyzed in some detail in Roorda and Schumacher (2007), in a more restrictive setting with finite outcome space. We currently finalize a working paper in which we have generalized the results to the setup with $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\dots,T}, P)$. We here mention two results that are crucial now. Firstly, under some mild conditions, sequential consistency implies conditional consistency. Secondly, under both forms of weak time consistency, and also under strong time consistency, the initial acceptability measure ϕ already completely determines all subsequent measures ϕ_t for $t = 1, \dots, T$, by the *update rule*

$$\phi_t(X) = \operatorname{ess\,sup}\{Y \in L_t^\infty \mid \phi((X - Y)\mathbb{1}_F) \geq 0 \text{ for all } F \in \mathcal{F}_t\}. \tag{3.2}$$

Notice that DTVaR as given above indeed satisfies this update rule. We remark that the rule itself is in fact obvious, and not new, cf. Tutsch (2006).

Both notions of weak time consistency are quite natural, not only in the context of pricing, with c close to 1, but also in a regulatory context, with c close to zero. We will now focus on the applying the strongest of the two, sequential consistency, on multiperiod TVaR.

4 How to apply TVaR in a sequentially consistent way?

In view of the results in Roorda and Schumacher (2007), it is not a surprise that a sequentially consistent extension of TVaR to a dynamic setting can be obtained as the update of the following initial measure.

(4.1)

$$\mathrm{STVaR}_c(X) := \inf\{E_P[X\Pi_{s=1}^T L_s] \mid L_s \in L_s^\infty, 0 \leq \Pi_{s'=s}^T L_{s'} \leq c^{-1}, E_P[L_s|\mathcal{F}_{s-1}] = 1 \text{ for } s = 1, \ldots, T\}.$$

Again, for later time, $\mathrm{STVaR}_{c,t}$ allows for a similar expression, with expectation now conditional on $\mathcal{F}_t$, and s starting in $t+1$. Notice the crucial difference with just taking TVaR_c over the entire period, which would amount to imposing the inequalities in (4.1) only for $s = T$. This is the key in avoiding the problem of extending TVaR to the dynamic case as signalled in Artzner et al. (2007)

It can be derived that STVaR is indeed sequentially consistent. Accumulation of conservatism is now avoided by construction, as only probability measures are taken into account having derivative at most c^{-1}, instead of c^{-T} in DTVaR. Notice that the definition does not involve any extra parameters, unlike the definition in Pflug (2007, Def. 3.29).

In the presentation we will illustrate that STVaR_c is really different from DTVaR or variants of it with time depending levels, by a simple example.

We will also pay attention to computational aspects. STVaR of course can no longer be computed by straightforward backward recursion, but we will describe a more subtle recursive scheme for STVaR. Based on these results, we will determine an expression of STVaR_c as a conditional expectation $E_P[X|B]$ with $P(B) = c$, which can also serve as a basis for calculating STVaR by Monte-Carlo simulation.

References

Artzner, Ph., F. Delbaen, J.-M. Eber, D. Heath, and H. Ku, 2007. Coherent multiperiod risk adjusted values and Bellman's principle. *Annals of Operations Research*, **152-1**, pp. 5-22.

Föllmer, H., and A. Schied, 2004. *Stochastic Finance. An introduction in discrete time* (2nd ed.). De Gruyter Studies in Mathematics, Vol. 27, de Gruyter, Berlin / New York.

Föllmer, H., and I. Penner, 2006. Convex risk measures and the dynamics of their penalty functions. *Statistics & Decisions*, **24**, 61–96.

Frittelli, M. and E. Rosazza Gianin, 2004. Dynamic convex risk measures. In: G. Szegö (ed.), *Risk Measures for the 21st Century*, Wiley, New York.

Pflug, G., 2007. *Modeling, Measuring, and Managing Risk.* World Scientific.

Roorda, B., and J. M. Schumacher, 2007. Time consistency conditions for acceptability measures, with an application to tail value at risk, to appear in *Insurance: Mathematics and Economics.*

Tutsch, S., 2006. Update rules for convex risk measures. Manuscript, Humboldt-Universität zu Berlin.