The effect of a safety-based incentive structure on route choice behaviour over time

A day-to-day dynamical system approach

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Contents

Abstract

This paper studies the route choice behaviour after the implementation of an incentive structure which rewards the travellers for following safe routes assigned to them. The routes that are considered safe are predetermined, so are the rewards. The effect of the incentive structure is simulated in a dynamic setting so as to analyse the consequences of its implementation. It also provides a modelling framework for future studies on dynamic incentive structure. This paper focuses on the adjustment in travellers' route choice behaviour as a result of the incentive structure implemented. Such adjustment is realised by a day-to-day learning process. This paper shows that the effect of introducing such incentive structures on the performance of the traffic network depends on the behaviour of the travellers as well as the incentive structures. We also demonstrate how sequential implementation of the incentive structure can ensure a smoother transition from the original flow pattern to the new, desired flow pattern.

Keywords

Route choice, traffic safety, incentive structure, day-to-day traffic dynamics

1 Introduction

Similar (but contrary) to the concept of road pricing, rewarding travellers for following safe routes can bring benefit from the system's point of view (Van Arem and Bie, 2008). In road pricing, travellers pay for what they cost on the system (e.g., increased congestion); under the incentive structure on safe driving, travellers get paid for following the assigned safe routes and thus reducing the accident rates in the system. The immediate consequences of implementing such an incentive structure are studied in this paper. We assume that the safe routes as well as the rewards have already been predetermined. When the incentive structure is implemented on the traffic network, the route costs (or route utilities) on a given origin-destination (OD) pair are changed as a result. Facing the changes, travellers intended to travel on this OD pair may reconsider their route choices. If the route choices are performed repeatedly over time (such as daily) after the implementation, the result is a time-evolutionary flow pattern on the traffic network.

The evolution of traffic flow over time as a result of the implemented incentive structure is captured by travellers' learning process. This is formulated as a day-to-day dynamical system, where the perceived cost of traversing a route on a given day is derived from the experienced costs in the previous days. The perceived costs are therefore updated after each day. Route choices on a day are then made based on the perceived costs of that day, resulting in time-evolutionary traffic flow. The stochastic equilibrium is achieved when the perceived costs (as well as the network flow) remain stationary (Bie, 2008). Stability of the equilibrium is an important issue because unstable equilibrium is unlikely to persist in reality. Besides stability, the attraction domain of the equilibrium is useful in cases of network modifications (such as the implementation of an incentive structure) in order to ensure the eventual convergence to the new equilibrium.

The stability of day-to-day traffic dynamics has been first addressed by Horowitz (1984), who showed that ever for a well-behaved rational system, the evolution over time may not converge to equilibrium. In the two-link network studied, the stochastic equilibrium is globally stable only if several strict conditions are met. Cantarella and Cascetta (1995), Watling (1999), Bie and Lo (2007a) and Bie (2008) extended Horowitz's work to the stability of stochastic equilibrium for general networks. Watling (1999) provided a method to estimate the attraction domain. Bie (2008) further showed how the exact range of the attraction domain can be determined. Bie and Lo (2006, 2007b) showed the possible consequences of network modifications. Some well-intended actions may lead to undesirable situations as a result of the dynamic adjustment process. Lo and Bie (2006) and Bie and Lo (2006) demonstrated the possibility of smoothening the transition from the original flow pattern to the desired pattern through a method called sequential implementation.

In this paper, the day-to-day traffic dynamics is formulated in Section 2, following by formulation of the incentive structure and its implementation in Section 3. We also provide some theoretical analysis on the possible consequences and several numerical examples. Section 4 concludes the paper with some remarks and discussions for future research.

2 Day-to-day traffic dynamics

The dynamical evolution of traffic from day to day is formulated as a dynamical system, characterised by a recurrence function of the vector of perceived route costs. Travellers update their perception on a daily basis. In the updated perceived costs, both previous perception and recent experience (of the actual travel costs) are taken into account. The steady state of this dynamical system is identical to stochastic user equilibrium in the static system. Therefore the steady state is a dynamic equilibrium and the dynamical evolution represents the process of pursuing equilibrium.

2.1 The dynamical system

Consider a network with N OD pairs. Each OD pair i ($i = 1, 2, ..., N$) is connected by a set of routes, denoted as $\mathbf{R}_i$, with $m_i = |\mathbf{R}_i|$ as the number of routes connecting the OD pair. The total number of OD routes for the whole network is given as $M = \sum_{i=1}^{N} m_i$. There M routes are numerated as $1, 2, ..., m_1$ for the m_1 routes in $\mathbf{R}_1$, as $m_1+1, m_1+2, ..., m_1+m_2$ for the m_2 routes in $\mathbf{R}_2$, …, and finally as $M-m_N+1, M-m_N+2, ..., M$ for the m_N routes in $\mathbf{R}_N$.

On day n, travellers' knowledge of the network is represented by the vector of mean perceived route costs, $\mathbf{C}^{(n)} = [C_1^{(n)}, C_2^{(n)}, ..., C_r^{(n)}, ..., C_M^{(n)}]^{\mathrm{T}}$. Travel demand for the day is a non-increasing function of the perceived cost,

$$\mathbf{d}^{(n)} = d(\mathbf{C}^{(n)}), \tag{1}$$

where the M-vector $\mathbf{d}^{(n)}$ is the transformed OD demand vector, in the following form:

$$\begin{aligned}\mathbf{d}^{(n)} = [&\underbrace{d_1^{(n)}, d_1^{(n)}, ..., d_1^{(n)}}_{m_1}, \underbrace{d_2^{(n)}, d_2^{(n)}, ..., d_2^{(n)}}_{m_2}, ..., \\ &\underbrace{d_i^{(n)}, d_i^{(n)}, ..., d_i^{(n)}}_{m_i}, ..., \underbrace{d_N^{(n)}, d_N^{(n)}, ..., d_N^{(n)}}_{m_N}]^T.\end{aligned} \tag{2}$$

Here $d_i^{(n)} \geq 0$ denotes the travel demand of OD pair i on day n.

Traffic is assigned according to logit route choice model. The probability of choosing a route is a function of the mean perceived route costs and the dispersion parameter θ ($\theta \geq 0$). For a traveller on OD pair i, the probability of choosing route r ($r \in \mathbf{R}_i$) under the perceived cost $\mathbf{C}^{(n)}$ is given as

$$\begin{aligned}\Pr\{r, \mathbf{C}^{(n)}\} &= \frac{\exp(-\theta C_r^{(n)})}{\sum_{s \in \mathbf{R}_i} \exp(-\theta C_s^{(n)})} \\ &= \frac{1}{\sum_{s \in \mathbf{R}_i} \exp[\theta(C_r^{(n)} - C_s^{(n)})]} \\ &= \frac{1}{1 + \sum_{s \in \mathbf{R}_i, s \neq r} \exp[\theta(C_r^{(n)} - C_s^{(n)})]}.\end{aligned} \tag{3}$$

The corresponding flow assignment is

$$\mathbf{f}^{(n)} = diag\{\mathbf{d}^{(n)}\}\mathbf{p}^{(n)}, \tag{4}$$

where $\mathbf{f}^{(n)} = [f_1^{(n)}, f_2^{(n)}, ..., f_r^{(n)}, ..., f_M^{(n)}]^{\mathrm{T}}$ gives the traffic flow on each route and the M -vector $\mathbf{p}^{(n)}$ is the choice probability vector, in the following form:

$$\begin{aligned}\mathbf{p}^{(n)} &= p(\mathbf{C}^{(n)}) \\ &= [\Pr\{1, \mathbf{C}^{(n)}\}, \Pr\{2, \mathbf{C}^{(n)}\}, ..., \Pr\{r, \mathbf{C}^{(n)}\}, ..., \Pr\{M, \mathbf{C}^{(n)}\}]^T.\end{aligned} \tag{5}$$

The actual traffic costs are then determined by the travel cost (or performance) functions:

$$\begin{aligned}\mathbf{c}^{(n)} &= c(\mathbf{f}^{(n)}) \\ &= [c_1(\mathbf{f}^{(n)}), c_2(\mathbf{f}^{(n)}), ..., c_r(\mathbf{f}^{(n)}), ..., c_M(\mathbf{f}^{(n)})]^{\mathrm{T}}.\end{aligned} \tag{6}$$

Travellers' perceived cost is updated on a daily basis. If the perceived route travel cost on a day is equal to the actual cost, we expect that the perceived cost after the update is the same as the perceived cost before the update. However, if the perceived cost and the actual cost are of different values, the updated perceived cost would be some place in between the two values. If a linear relationship is presumed, we have

$$\mathbf{C}^{(n)} = \beta \mathbf{c}^{(n-1)} + (1-\beta)\mathbf{C}^{(n-1)}, \tag{7}$$

where parameter $\beta \in [0,1]$ represents the forgetfulness of the travellers, or their activeness in absorbing new information. For the extreme case of $\beta = 1$, memory of past knowledge is discarded; on the other hand, $\beta = 0$ means that the new information of actual travel cost is not used in the knowledge updating. Rational behaviour would take the value of β in between, i.e. $0 < \beta < 1$. The bigger value β is, the more travellers rely on new information.

In (7), the updated perceived cost for day $n-1$ is taken as the perceived cost for day n. Travel demand for day n is determined by the demand function (1). Traffic is assigned according to (4) and the actual travel cost is then given by (6). By updating the perceived cost on day n with the actual cost on day n, the perceived cost for day $n+1$ is generated. Now a full circle of travellers' learning process is finished, as shown in Figure 1. A dynamical system is formed by combining (1), (4), (6), and (7):

$$\begin{cases}\mathbf{d}^{(n)} = d(\mathbf{C}^{(n)}), \\ \mathbf{f}^{(n)} = diag\{\mathbf{d}^{(n)}\} p(\mathbf{C}^{(n)}), \\ \mathbf{c}^{(n)} = c(\mathbf{f}^{(n)}), \\ \mathbf{C}^{(n+1)} = \beta \mathbf{c}^{(n)} + (1-\beta)\mathbf{C}^{(n)}.\end{cases} \tag{8}$$

Since demand, flow, and actual cost can be subsequently determined as soon as the perceived cost is known, we can simplify the dynamical system (8) to the following form:

$$\mathbf{C}^{(n+1)} = \beta c[diag\{d(\mathbf{C}^{(n)})\} p(\mathbf{C}^{(n)})] + (1-\beta)\mathbf{C}^{(n)}. \tag{9}$$

Here the perceived cost alone is enough to represent the dynamical system. The evolution of the system is characterized by the recurrence function (9) of perceived cost. This recurrence function maps the perceived cost on a day to the perceived cost on the next day.

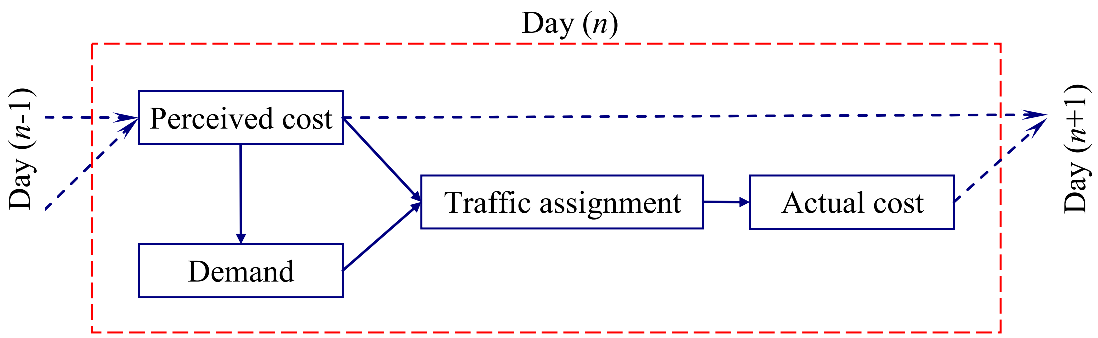

Figure 1: Day-to-day traffic dynamics

For the special case of fixed (or inelastic) demand, what matters is not the absolute values of the perceived travel costs but the cost differences between alternative routes on an OD pair (cf. (3)). The system has then only $M-N$ dimensions. Consider now that travel demand is fixed and independent of time (i.e. day n),

$$\begin{aligned}\mathbf{d} = [&\underbrace{d_1, d_1, \ldots, d_1}_{m_1}, \underbrace{d_2, d_2, \ldots, d_2}_{m_2}, \ldots, \\ &\underbrace{d_i, d_i, \ldots, d_i}_{m_i}, \ldots, \underbrace{d_N, d_N, \ldots, d_N}_{m_N}]^T.\end{aligned} \tag{10}$$

The $M-N$-vector of cost differences can be given in the following form:

$$\begin{aligned}\mathbf{g}_{M-N}{}^{(n)} &= [g_1^{(n)}, g_2^{(n)}, \ldots, g_{m_1-1}^{(n)}; \\ &\quad g_{m_1}^{(n)}, g_{m_1+1}^{(n)}, \ldots, g_{m_1+m_2-2}^{(n)}; \ldots; \\ &\quad g_{M-N-m_N+2}^{(n)}, g_{M-N-m_N+3}^{(n)}, \ldots, g_{M-N}^{(n)}]^T \\ &= [C_1^{(n)} - C_2^{(n)}, C_1^{(n)} - C_3^{(n)}, \ldots, C_1^{(n)} - C_{m_1}^{(n)}; \\ &\quad C_{m_1+1}^{(n)} - C_{m_1+2}^{(n)}, C_{m_1+1}^{(n)} - C_{m_1+3}^{(n)}, \ldots, C_{m_1+1}^{(n)} - C_{m_1+m_2}^{(n)}; \ldots; \\ &\quad C_{M-m_N+1}^{(n)} - C_{M-m_N+2}^{(n)}, C_{M-m_N+1}^{(n)} - C_{M-m_N+3}^{(n)}, \ldots, C_{M-m_N+1}^{(n)} - C_M^{(n)}]^T.\end{aligned} \tag{11}$$

Here each OD pair i has $m_i - 1$ entries of cost differences between alternative routes. For the convenience in calculating route choice probabilities, we can transform the above $M-N$-vector into an M-vector by adding 0 entries to it,

$$\begin{aligned}\mathbf{g}_M{}^{(n)} &= [0, g_1^{(n)}, g_2^{(n)}, \ldots, g_{m_1-1}^{(n)}; \\ &\quad 0, g_{m_1}^{(n)}, g_{m_1+1}^{(n)}, \ldots, g_{m_1+m_2-2}^{(n)}; \ldots; \\ &\quad 0, g_{M-N-m_N+2}^{(n)}, g_{M-N-m_N+3}^{(n)}, \ldots, g_{M-N}^{(n)}]^T.\end{aligned} \tag{12}$$

For simplicity reasons the notation of $\mathbf{g}^{(n)}$ may be used in place of both $\mathbf{g}_{M-N}{}^{(n)}$ and $\mathbf{g}_M{}^{(n)}$ if no confusion is likely to arise. The route choice probability can then be written as

$$\Pr\left\{r, \mathbf{g}^{(n)}\right\} = \frac{1}{\sum_{s \in \mathbf{R}_i} \exp[\theta(g_{Ms}{}^{(n)} - g_{Mr}{}^{(n)})]}, \forall r \in \mathbf{R}_i. \tag{13}$$

Traffic dynamics from day to day can then be represented by the recurrence function of the cost difference vector,

$$\mathbf{g}^{(n+1)} = \beta c_g[diag\{\mathbf{d}^{(n)}\} p(\mathbf{g}^{(n)})] + (1-\beta)\mathbf{g}^{(n)}, \tag{14}$$

where $\mathbf{c}_g{}^{(n)}$ is a transformation of $\mathbf{c}^{(n)}$, in the way similar to (11) or (12).

2.2 The dynamic equilibrium

We consider the dynamical system in the generic form of

$$\mathbf{x}^{(n+1)} = f(\mathbf{x}^{(n)}), \tag{15}$$

where f is the recurrence function and $\mathbf{x}^{(n)}$ represents the state of the dynamical system on day n. For day-to-day traffic dynamics, $\mathbf{x}^{(n)}$ can be $\mathbf{C}^{(n)}$ or $\mathbf{g}^{(n)}$. Here $\mathbf{x}^{(n+1)}$ is the *image* of $\mathbf{x}^{(n)}$ and $\mathbf{x}^{(n)}$ is a *pre-image* of $\mathbf{x}^{(n+1)}$. Once the initial state (or initial point) $\mathbf{x}^{(0)}$ is given, any future state can be derived by iterating the recurrence function,

$$\mathbf{x}^{(n)}\Big|\mathbf{x}^{(0)} = \underbrace{f(f(...f}_{n}(\mathbf{x}^{(0)})...)) = f^{(n)}(\mathbf{x}^{(0)}). \tag{16}$$

The notation $\mathbf{x}^{(n)}\Big|\mathbf{x}^{(0)}$ implies that the state on day n ($n = 1, 2, ...$) intrinsically depends on the initial state. The specification of the initial state can be omitted if no confusion is likely to arise. The *trajectory* of the dynamical evolution starting from $\mathbf{x}^{(0)}$ is

$$\begin{aligned}&\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, ..., \mathbf{x}^{(n)}, ...\\&= \mathbf{x}^{(0)}, f(\mathbf{x}^{(0)}), f^{(2)}(\mathbf{x}^{(0)}), ..., f^{(n)}(\mathbf{x}^{(0)}), ...\end{aligned} \tag{17}$$

If the sequence in (17) has a limit (i.e. the sequence converges), then this limit gives a steady (or stationary) state of the dynamical system. Such a steady state $\mathbf{x}^*$ is the solution of

$$\mathbf{x}^* = f(\mathbf{x}^*), \tag{18}$$

i.e. it is a fixed point of the recurrence function. By repeatedly applying the recurrence function, we have

$$\underbrace{f(f(...f}_{n}(\mathbf{x}^*)...)) = \mathbf{x}^*, n = 1, 2, ... \tag{19}$$

This means that a dynamical evolution starting from the fixed point will forever remain at that point (hence 'stationary').

Consider the recurrence function of day-to-day traffic dynamics, (9), its fixed point is any perceived cost vector that satisfies

$$\mathbf{C}^* = \beta c[diag\{d(\mathbf{C}^*)\}p(\mathbf{C}^*)] + (1-\beta)\mathbf{C}^*. \tag{20}$$

If β equals 0 then any feasible cost vector will be a fixed point. This is not realistic but it makes sense because $\beta = 0$ actually means drivers never update their knowledge (and therefore never bother to change their route choices, resulting in stationary traffic flow). In most realistic cases, β is positive and (20) is equivalent to

$$\mathbf{C}^* = c[diag\{d(\mathbf{C}^*)\}p(\mathbf{C}^*)], \tag{21}$$

which means that the (mean) perceived cost is identical to the actual cost. Therefore a fixed point of (9) is also a stochastic user equilibrium of the static system. The dynamical evolution as represented by (9) signifies the process of achieving equilibrium. The same holds for the case of fixed demand, where $\mathbf{C}^*$ is replaced by $\mathbf{g}^*$.

2.3 Equilibrium attainability and attraction domain

The equilibrium $\mathbf{x}^*$ is *attainable* from an initial point $\mathbf{x}^{(0)}$ if

$$\lim_{n\to\infty} \mathbf{x}^{(n)} \Big| \mathbf{x}^{(0)} = \mathbf{x}^*. \tag{22}$$

We denoted the attainability by $\mathbf{x}^{(0)} \square \mathbf{x}^*$. The equilibrium is always attainable from itself, i.e. $\mathbf{x}^* \square \mathbf{x}^*$. If $\mathbf{x}^*$ is attainable from every point in the set $\mathbf{S}$, we can also say that $\mathbf{x}^*$ is attainable from $\mathbf{S}$, denoted as $\mathbf{S} \square \mathbf{x}^*$.

The *attraction domain* (or *attraction basin*) for the equilibrium $\mathbf{x}^*$, denoted as $\mathbf{B}(\mathbf{x}^*)$, is the set of all points that $\mathbf{x}^*$ is attainable from:

$$\mathbf{B}(\mathbf{x}^*) = \{\mathbf{x}^{(0)} : \mathbf{x}^{(0)} \square \mathbf{x}^*\}. \tag{23}$$

Obviously, $\mathbf{x}^* \in \mathbf{B}(\mathbf{x}^*)$ and $\mathbf{B}(\mathbf{x}^*) \square \mathbf{x}^*$.

Example 1 (Watling, 1999) Consider a network with one OD pair connected by three routes. Demand is fixed at 2 units. The cost functions for the three routes are given as:

$$c_1(\mathbf{f}) = f_1 + 3f_2 + 1,$$
$$c_2(\mathbf{f}) = 2f_1 + f_2 + 2,$$
$$c_3(\mathbf{f}) = f_3 + 6.$$

The dispersion parameter is $\theta = 1$ and the updating ratio is $\beta = 0.2$.

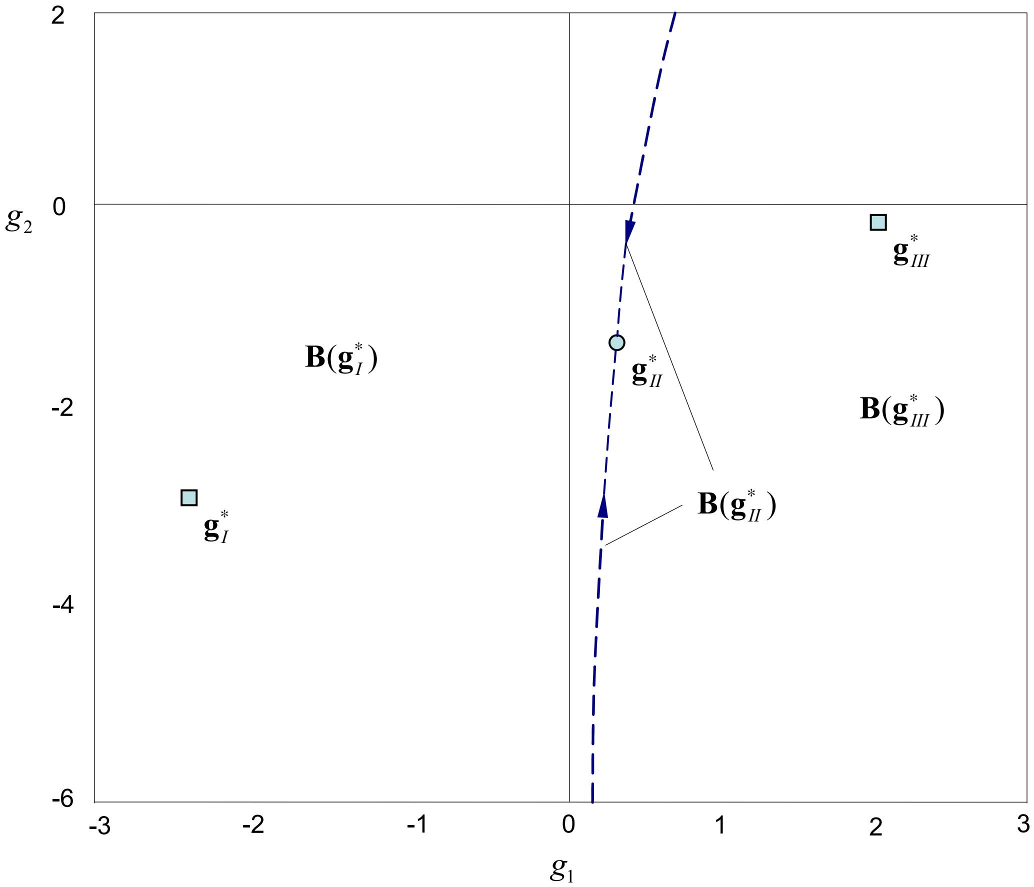

Figure 2: Attraction domains

There are multiple equilibria. All system evolution converges to one of the following three equilibria:

$$\mathbf{f}_I^* = [1.752, 0.151, 0.097]^{\mathrm{T}};$$

$$\mathbf{f}_{II}^* = [0.768, 1.031, 0.201]^{\mathrm{T}};$$

$$\mathbf{f}_{III}^* = [0.226, 1.588, 0.186]^{\mathrm{T}},$$

or, in cost differences, given as

$$(g_1, g_2)_I^* = (-2.449, -2.892);$$

$$(g_1, g_2)_{II}^* = (0.292, -1.341);$$

$$(g_1, g_2)_{III}^* = (1.951, -0.195).$$

The attraction domains of the three equilibria are shown in Figure 2. All points on the dashed curve converge to $\mathbf{g}_{II}^*$. This curve represents the attraction domain of $\mathbf{g}_{II}^*$, which is an unstable equilibrium point. The region left to the curve gives the attraction domain of $\mathbf{g}_I^*$, while the region on the right is the attraction domain of $\mathbf{g}_{III}^*$. The phase portrait in Figure 3 shows how system evolutions are attracted to the equilibrium. ◊

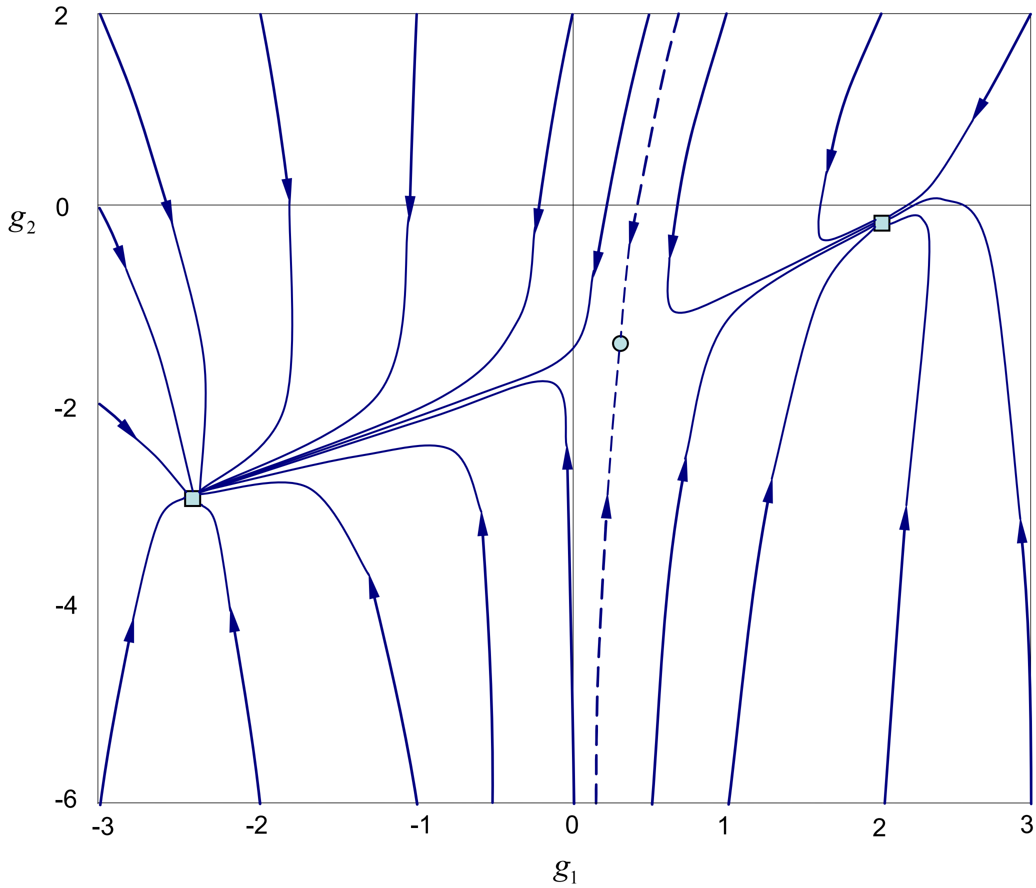

Figure 3: Phase portrait

3 The incentive structure

The incentive structure describes how the rewards for following safe routes are distributed. It includes both the selection (identification) of safe routes and the amount of reward for each safe route. This section formulates the incentive structure and briefly deliberates the consequences of its implementation.

3.1 Rewards for following safe routes

Consider the same network as in Section 2. The incentive structure is represented by the M -vector of route following rewards,

$$\mathbf{I}^{(n)} = [I_1^{(n)}, I_2^{(n)}, ..., I_r^{(n)}, ..., I_M^{(n)}]^{\mathrm{T}}, \tag{24}$$

where all entries are non-negative. If $I_r^{(n)} > 0$ then a traveller is rewarded the amount of $I_r^{(n)}$ for following the route r on day n. If $I_r^{(n)} = 0$ then there is no reward for choosing route r on day n. For simplistic cases the incentive structure is constant over time. Then we can remove the superscript n.

Suppose the incentive structure is published to all travellers. The travel demand, if elastic, will be affected. Because the incentives (rewards) are gains rather than costs, they increase the utility of safe routes and also the total travel demand. The new demand function is

$$\mathbf{d}^{(n)} = d(\mathbf{C}^{(n)} - \mathbf{I}^{(n)}). \tag{25}$$

Under the incentive structure, the probability that a traveller on OD pair i will choose route r ($r \in \mathbf{R}_i$) on day n is given as

$$\Pr\left\{r, \mathbf{C}^{(n)}, \mathbf{I}^{(n)}\right\} = \frac{1}{1 + \sum_{s \in \mathbf{R}_i, s \neq r} \exp[\theta(C_r^{(n)} - I_r^{(n)} - C_s^{(n)} + I_s^{(n)})]}. \tag{26}$$

Compare (26) with (3) and we can see that the incentive structure will attract more travellers to the routes with a positive reward (i.e. safe routes).

3.2 Implementation of the incentive structure

Suppose that before implementing the incentive structure the traffic system is already at equilibrium (i.e. stationary network flow from day to day), then the implementation will disequilibrate the system. The resulting network flow will no longer be stationary over time. Mostly it will dynamically evolve to a new equilibrium point. However, it is also possible that the system is perpetually disturbed and never converges to an equilibrium solution.

Suppose that before the incentive structure is implemented, the system is at equilibrium $\mathbf{x}^*$. After the implementation, the new equilibrium solution is $\mathbf{x}_\mathrm{n}^*$, with attraction domain $\mathbf{B}(\mathbf{x}_\mathrm{n}^*)$. If

$$\mathbf{x}^* \in \mathbf{B}(\mathbf{x}_\mathrm{n}^*), \tag{27}$$

as shown in Figure 4, then the new equilibrium is attainable from the original equilibrium. However, if

$$\mathbf{x}^* \notin \mathbf{B}(\mathbf{x}_\mathrm{n}^*), \tag{28}$$

as shown in Figure 5, then the new equilibrium is unattainable from the original equilibrium.

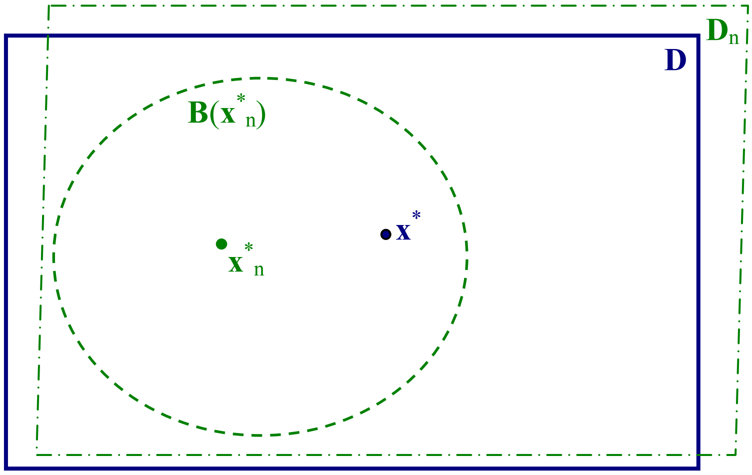

Figure 4: Attainable new equilibrium from the original equilibrium

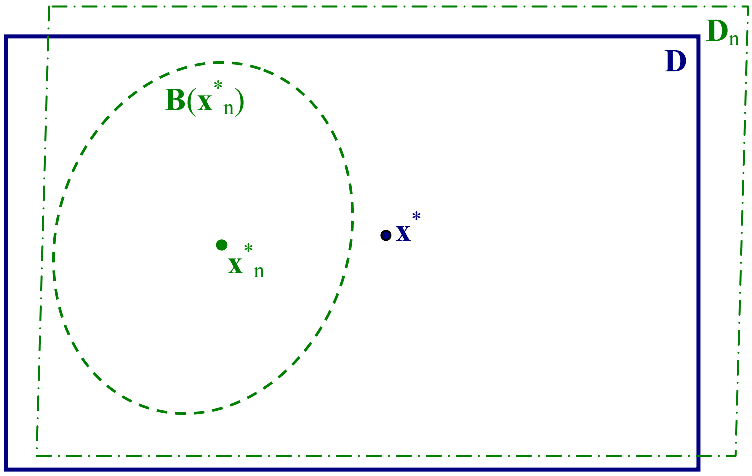

Figure 5: Unattainable new equilibrium from the original equilibrium

Example 2 Consider a two-link network with cost functions given as

$$c_1 = 4\left[1 + 0.15\left(\frac{f_1}{1000}\right)^4\right], c_2 = 3.5\left[1 + 0.15\left(\frac{f_2}{600}\right)^4\right].$$

Demand is elastic and is as given:

$$d = f_1 + f_2 = 2000 - 100 \cdot \min\{C_1 - I_1, C_2 - I_2\}, d \geq 0.$$

The dispersion parameter in the logit choice model is $\theta = 2$ and the updating ratio is $\beta = 0.3$. For the current system without incentive structure, i.e. $\mathbf{I} = 0$, the following equilibrium has been established:

$$(C_1^*, C_2^*) = (4.342, 4.453), (f_1^*, f_2^*) = (896, 696).$$

If the incentive structure $\mathbf{I} = [0, 0.1]^{\mathrm{T}}$ is implemented on day 0, the subsequent system evolution is shown in Figure 6, which gradually converges to the new equilibrium:

$$\mathbf{C}_{\mathrm{n}}^* - \mathbf{I} = [4.326, 4.422]^{\mathrm{T}}, \mathbf{f}_{\mathrm{n}}^* = [859, 709]^{\mathrm{T}}.$$

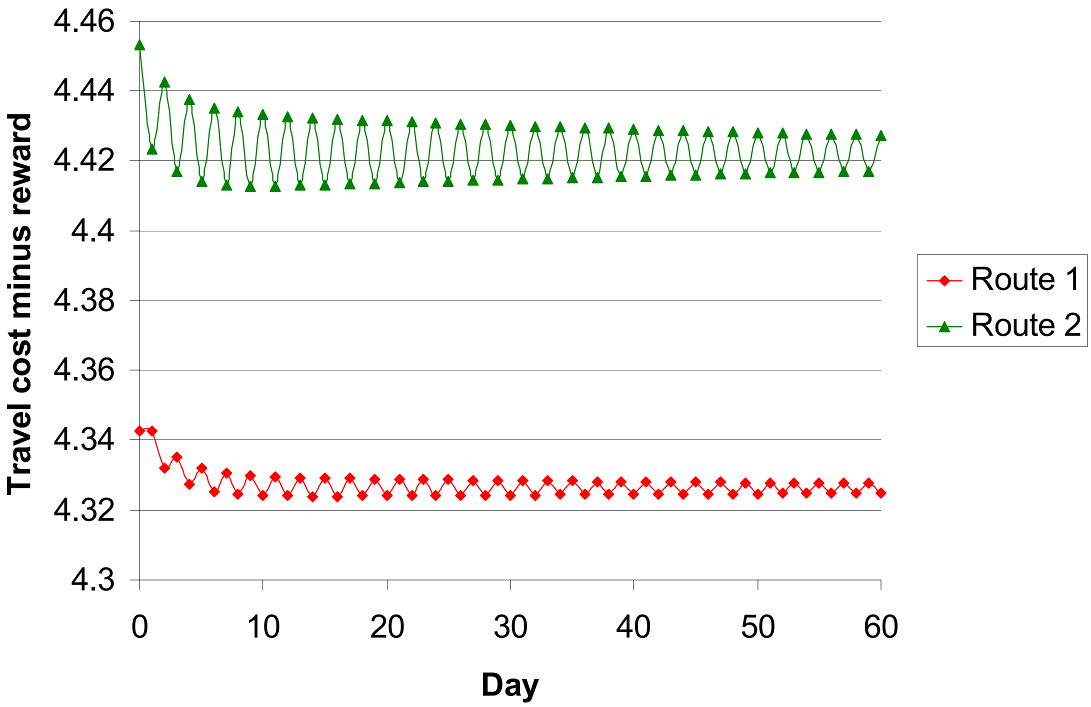

Figure 6: Attainable new equilibrium under $I=[0,0.1]^T$

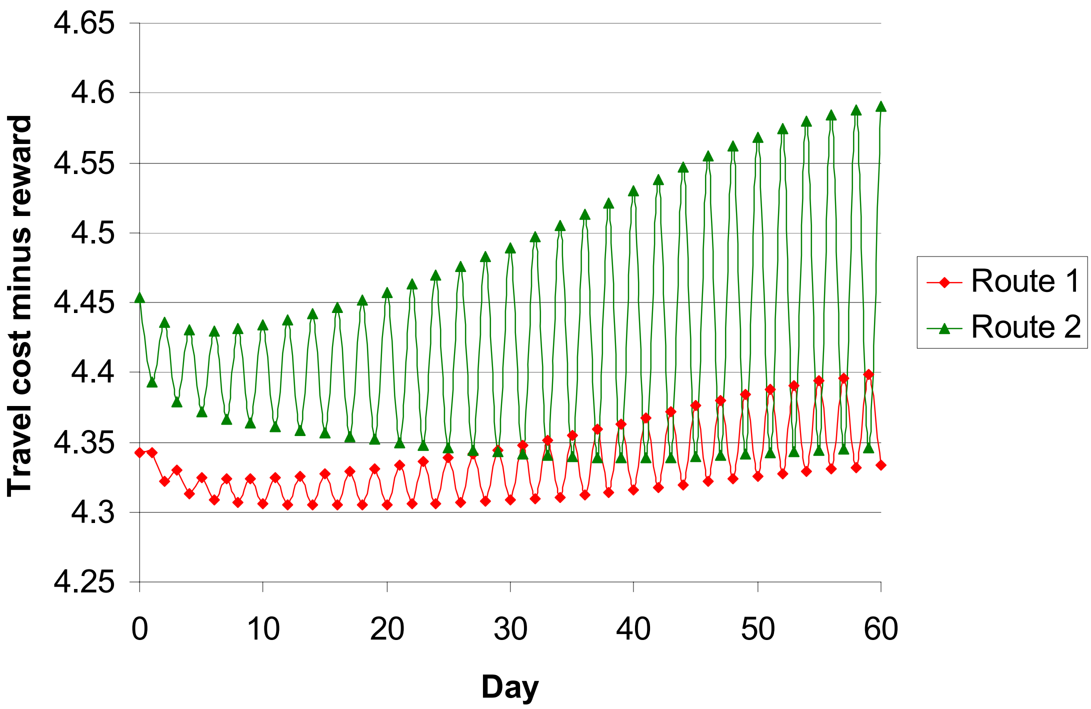

Figure 7: Unattainable new equilibrium under $I=[0,0.2]^T$

However, if the incentive structure $\mathbf{I} = [0, 0.2]^{\mathrm{T}}$ is implemented on day 0, the subsequent system evolution is shown in Figure 7. The evolution does not converge to equilibrium, although a new equilibrium does exist:

$$\mathbf{C}_{\mathrm{n}}^{*} - \mathbf{I} = [4.311, 4.392]^{\mathrm{T}}, \mathbf{f}_{\mathrm{n}}^{*} = [848, 721]^{\mathrm{T}}. \Diamond$$

3.3 Sequential implementation of the incentive structure

The problem of unattainable new equilibrium can be solved by a method called *sequential implementation*. This means that the incentive structure is not directly implemented. Instead, a sequence of incentive structures is implemented one after one, ending at the planned incentive structure. In each step, a smooth transition is ensured from the (step-wise) original equilibrium to the (step-wise) new equilibrium.

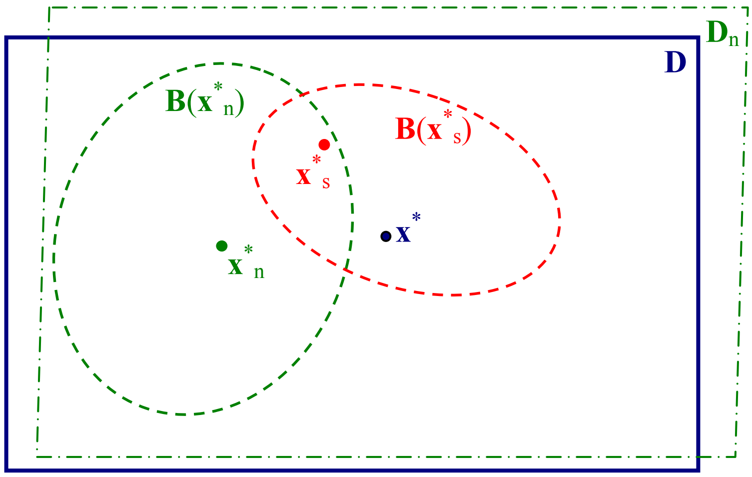

Figure 8: Sequential implementation

Consider the most simplistic case where the sequence has only two steps. As shown in Figure 8, we plan to introduce an incentive structure such that the original equilibrium $\mathbf{x}^{*}$ is transferred to the new equilibrium $\mathbf{x}_{\mathrm{n}}^{*}$. However, because $\mathbf{x}^{*} \notin \mathbf{B}(\mathbf{x}_{\mathrm{n}}^{*})$, in the new network the dynamical evolution starting from $\mathbf{x}^{*}$ will not converge to $\mathbf{x}_{\mathrm{n}}^{*}$. Nevertheless, such convergence can be achieved by introducing a transitional step (or a sequence of steps).

Before implementing the planned incentive structure, we can implement a transitional incentive structure after whose implementation the system equilibrium is $\mathbf{x}_{\mathrm{s}}^{*}$ with attraction domain $\mathbf{B}(\mathbf{x}_{\mathrm{s}}^{*})$. As long as the following two conditions are satisfied:

$$\mathbf{x}^{*} \in \mathbf{B}(\mathbf{x}_{\mathrm{s}}^{*}), \tag{29}$$

$$\mathbf{x}_{\mathrm{s}}^{*} \in \mathbf{B}(\mathbf{x}_{\mathrm{n}}^{*}), \tag{30}$$

we can achieve the $\mathbf{x}_{\mathrm{n}}^{*}$ through a step-by-step implementation of modifications. That is, we first change the network such that $\mathbf{x}^{*}$ is smoothly transferred to $\mathbf{x}_{\mathrm{s}}^{*}$; we then change the network such that $\mathbf{x}_{\mathrm{s}}^{*}$ is smoothly transferred to $\mathbf{x}_{\mathrm{n}}^{*}$.

It should be noted that such a transitional incentive structure does not always exist. It may then be worthwhile to search for a sequence of transitional incentive structures which would fulfil our purpose.

Example 3 Consider the same network as in Example 1. However, the current network has cost function on route 2 as:

$$c_2(\mathbf{f}) = 2f_1 + f_2 + 2.2.$$

Then there is a unique equilibrium in the system:

$$\mathbf{f}^* = [1.795, 0.112, 0.093]^{\mathrm{T}}, \mathbf{g}^* = [-2.770, -2.961]^{\mathrm{T}}.$$

Suppose that under the current equilibrium route 1 is considered unsafe and route 2 safe. The management wants to attract more traffic to route 2. The following incentive structure is to be implemented:

$$\mathbf{I} = [0, 0.2, 0]^{\mathrm{T}},$$

in the hope that after the implementation the network is exactly the same as in Example 1 and the equilibrium $\mathbf{f}^*_{III} = [0.226, 1.588, 0.186]^{\mathrm{T}}$ would be achieved.

However, if this incentive structure is directly applied, the dynamical evolution from the original equilibrium $\mathbf{g}^*$ converges to $\mathbf{g}^*_I$ rather than $\mathbf{g}^*_{III}$, as shown by the dashed curve in Figure 9.

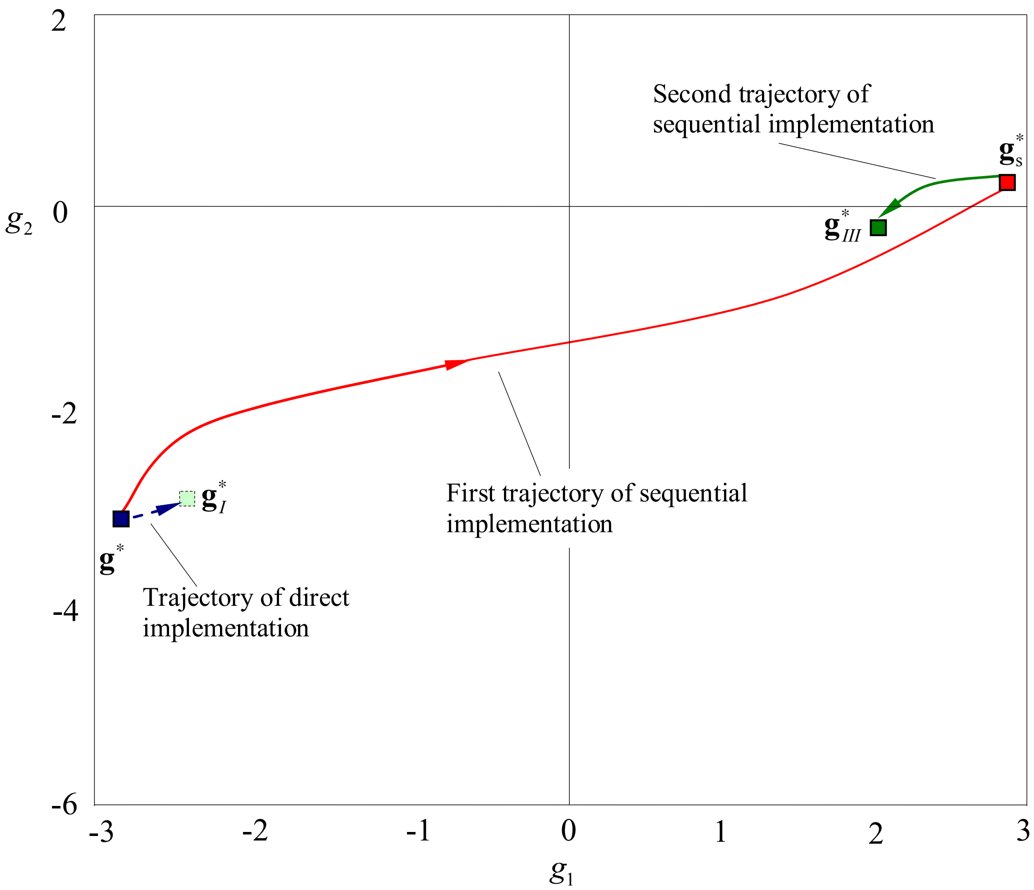

Figure 9: Sequential implementation of the incentive structure to achieve the desired equilibrium

To achieve $\mathbf{g}_{III}^{*}$ nevertheless, we can introduce the transitional incentive structure

$$\mathbf{I}_{s} = [0, 0.6, 0]^{\mathrm{T}}.$$

After its implementation the system evolution from $\mathbf{g}^{*}$ converges to

$$\mathbf{f}_{s}^{*} = [0.106, 1.759, 0.135]^{\mathrm{T}}, \mathbf{g}_{s}^{*} = [2.812, 0.247]^{\mathrm{T}}.$$

Once $\mathbf{g}_{s}^{*}$ is achieved, we can abandon the transitional incentive structure and reintroduce the planned incentive structure $\mathbf{I} = [0, 0.2, 0]^{\mathrm{T}}$. Then the system evolution from $\mathbf{g}_{s}^{*}$ converges to $\mathbf{g}_{III}^{*}$. This process is shown by the solid curves in Figure 9. ◊

4 Conclusions and discussions

This paper studied the consequences of implementing a safety-based incentive structure through a day-to-day dynamic setting of traffic flows. The incentive structure introduced disequilibrates the original system and brings a new equilibrium solution which is not automatically achieved. Instead, travellers adjust their route choices in response to the incentive structure implemented. This adjustment process can either lead to the new equilibrium or result in an ever-evolving non-stationary flow pattern. It is therefore important to ensure the attainability of the new equilibrium before implementing the incentive structure. When the original flow equilibrium is outside the new equilibrium's attraction basin, the method of sequential implementation can be adopted to attain the new equilibrium transitionally.

The learning process described in this paper is rather hypothetical and the numerical examples are quite simplistic. There is no evidence that real world networks behave in the same way as formulated here. However, this paper shows the possibilities of system evolution under the influence of the incentive structure introduced. The model can be made more realistic by relaxing the assumptions made in the formulation, such as the homogeneity of all users across all times. Another possible extension is the inclusion of departure time choice simultaneous to route choice.

We have not discussed the incentive structure in detail, particularly because we want to make the model as general as possible. In this way the model can be easily adapted for studies on incentive structures that are not based on safety, such as environmental reasons. The same model can also be used to address the effects of any 'added features' to the network.

Although the incentive structure has been formulated in a dynamic nature (with the time component), the numerical examples have been restricted to static incentive structures. The relaxation of this will allow traffic management more flexibility in optimising the incentive structure (e.g., by differentiating the time of day and/or over the days). Such flexibility may also be useful in the implementation of new incentive structures, as demonstrated by the method of sequential implementation to ensure (transitional) attainability of the new equilibrium.

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