User Equilibrium in Day-to-Day Traffic Dynamics: Stability, Attainability and Attraction Domain

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Abstract

This paper formulates the traffic assignment problem from a dynamical system approach. User equilibrium is realised as the steady state in day-to-day traffic dynamics. The dynamic equilibrium solution is shown to be identical to the equilibrium solution in a static model. The advantage of this dynamic approach is that it enables the analysis on equilibrium stability and attraction domains, as well as how disequilibrium states evolve towards equilibrium. These are important issues for the identification of the prevailing equilibrium in the long run, and for dynamic traffic management. This paper shows how the attraction domain can be determined or estimated. Moreover, the simultaneous route and departure time choice problem in the day-to-day setting is also formulated.

Keywords: day-to-day traffic dynamics, stability, attainability, attraction domain, simultaneous route and departure time choice

1 Introduction

Travellers who make repeated daily trips, such as commuters, do not choose the same route everyday even if the network remains unchanged over time [1]. This can be simulated by an updating process where travellers absorb new knowledge of the network on each day and then, according to this up-to-date knowledge, adjust their route choices for the next day's trip. This time-evolutionary process results in network flows that vary from day to day. The dynamic equilibrium is achieved when a steady flow pattern takes place in the network [2, 3].

If the updating process is rational and travellers follow the same principle of choosing the route with the least perceived cost, then the dynamic equilibrium solution is shown to be identical to the stochastic user equilibrium solution in the static model [4]. The dynamic model therefore characterises the day-to-day process in pursuit of equilibrium [5]; in terms of computation, it can also be adopted as algorithm for equilibrium solutions.

The advantage of the dynamic method is that it enables the studies on stability and attainability [2, 3, 4]. Stability is important in the sense that unstable equilibrium cannot sustain perturbations and therefore is unlikely to prevail in the long run [6]. For stable equilibrium, perturbations within a local bound will not cause the system diverging from equilibrium. As for attainability, the equilibrium is said to be attainable from a given disequilibrium state if the system evolution originated from this state will converge to the equilibrium; otherwise the equilibrium is unattainable from this state. The study on attainability has useful implications for the implementation of road pricing [7] and dynamic traffic management [8, 9].

This paper focuses on the attraction domain of equilibrium, providing characterisations and methods for estimation. The equilibrium's attraction domain consists of all the initial states which the equilibrium is attainable from. The ideal case is that this attraction domain covers the whole state space, which means that the equilibrium is globally attainable. However, in cases of multiple equilibria, the state space is divided into the attraction domains of different equilibrium solutions. If we know the exact range of these attraction domains, the question which equilibrium will prevail from a given initial state can then be answered by looking at the attraction domain that contains this initial state.

Theoretical analysis shows that the attraction domain of a stable equilibrium is always topologically open. Moreover, the boundary of this attraction domain consists of the evolution trajectories towards unstable equilibrium. Therefore the attraction domains of stable equilibria can be determined by looking at the trajectories to unstable equilibria. This also implies that if the system has only two equilibrium solutions, then one of them must be unstable. On the other hand, if we have two stable equilibrium points, on the line segment connecting these two points there must be at lease a point that converges to an unstable equilibrium.

Two methods are provided for estimating the attraction domain, where a subset of the domain is generated. The first method uses the Lyapunov function and searches for an invariant subset. The method is accurate yet rather conservative. In the second method, the estimate is determined by simulating the evolution from a number of sample states. This method is more efficient but may also introduce estimation errors. The errors can be reduced by improving the precision in sampling but cannot be completely eliminated. We will discuss the use of each of these two methods for estimating the attraction domain and demonstrate them with examples.

Furthermore, we will apply this day-to-day dynamical system approach for the case of simultaneous route and departure time choices. The objective is to study the departure time choices of travellers and their resultant effects on the overall network congestion. The question is with the choice of departure time incorporated, or an additional degree of freedom introduced, would the network performance exhibit a higher level of stability or attainability? Answer to this question is important for extending the day-to-day dynamical system approach to study more realistic travel choice problems.

2 Formulation of Day-to-Day Traffic Dynamics

The dynamical evolution of traffic from day to day is formulated as a dynamical system, characterised by a recurrence function of the vector of perceived route costs. Travellers update their perception on a daily basis. In the updated perceived costs, both previous perception and recent experience (of the actual travel costs) are taken into account. The steady state of this dynamical system is identical to stochastic user equilibrium in the static system. Therefore the steady state is a dynamic equilibrium and the dynamical evolution represents the process of pursuing equilibrium.

2.1 The dynamical system

Consider a network with N OD pairs. Each OD pair i ($i=1,2,...,N$) is connected by a set of routes, denoted as $\mathbf{R}_i$, with $m_i=|\mathbf{R}_i|$ as the number of routes connecting the OD pair. The total number of OD routes for the whole network is then given as $M=\sum_{i=1}^{N} m_i$. These M routes are numerated as $1,2,...,m_1$ for the m_1 routes in $\mathbf{R}_1$, as $m_1+1,m_1+2,...,m_1+m_2$ for the m_2 routes in $\mathbf{R}_2$, …, and as $M-m_N+1,M-m_N+2,...,M$ for the m_N routes in $\mathbf{R}_N$.

On day n, travellers' knowledge of the network is represented by the vector of mean perceived route costs, $\mathbf{C}^{(n)}=[C_1^{(n)},C_2^{(n)},...,C_r^{(n)},...,C_M^{(n)}]^{\mathrm{T}}$. Travel demand for the day is a non-increasing function of the perceived cost,

$$\mathbf{d}^{(n)}=d(\mathbf{C}^{(n)}), \tag{1}$$

where the M-vector $\mathbf{d}^{(n)}$ is the transformed OD demand vector, in the following form:

$$\begin{aligned}\mathbf{d}^{(n)}=[&\underbrace{d_1^{(n)},d_1^{(n)},...,d_1^{(n)}}_{m_1},\underbrace{d_2^{(n)},d_2^{(n)},...,d_2^{(n)}}_{m_2},...,\\ &\underbrace{d_i^{(n)},d_i^{(n)},...,d_i^{(n)}}_{m_i},...,\underbrace{d_N^{(n)},d_N^{(n)},...,d_N^{(n)}}_{m_N}]^T.\end{aligned} \tag{2}$$

Here $d_i^{(n)}\geq 0$ denotes the travel demand of OD pair i on day n.

Traffic is assigned according to logit route choice model. The probability of choosing a route is a function of the mean perceived route costs and the dispersion parameter θ ($\theta\geq 0$). For a traveller on OD pair i, the probability of choosing route r ($r\in\mathbf{R}_i$) under the perceived cost $\mathbf{C}^{(n)}$ is given as

$$\Pr\left\{r,\mathbf{C}^{(n)}\right\}=\frac{\exp(-\theta C_r^{(n)})}{\sum_{s\in\mathbf{R}_i}\exp(-\theta C_s^{(n)})}=\frac{1}{1+\sum_{s\in\mathbf{R}_i,s\neq r}\exp[\theta(C_r^{(n)}-C_s^{(n)})]}. \tag{3}$$

The corresponding flow assignment is

$$\mathbf{f}^{(n)}=diag\{\mathbf{d}^{(n)}\}\mathbf{p}^{(n)}, \tag{4}$$

where $\mathbf{f}^{(n)}=[f_1^{(n)},f_2^{(n)},...,f_r^{(n)},...,f_M^{(n)}]^{\mathrm{T}}$ gives the traffic flow on each route and the M-vector $\mathbf{p}^{(n)}$ is the choice probability vector, in the following form:

$$\mathbf{p}^{(n)}=p(\mathbf{C}^{(n)})=[\Pr\left\{1,\mathbf{C}^{(n)}\right\},\Pr\left\{2,\mathbf{C}^{(n)}\right\},...,\Pr\left\{r,\mathbf{C}^{(n)}\right\},...,\Pr\left\{M,\mathbf{C}^{(n)}\right\}]^T. \tag{5}$$

The actual traffic costs are then determined by the travel cost (or performance) functions:

$$\mathbf{c}^{(n)}=c(\mathbf{f}^{(n)})=[c_1(\mathbf{f}^{(n)}),c_2(\mathbf{f}^{(n)}),...,c_r(\mathbf{f}^{(n)}),...,c_M(\mathbf{f}^{(n)})]^{\mathrm{T}}. \tag{6}$$

Travellers' perceived cost is updated on a daily basis. If the perceived route travel cost on a day is equal to the actual cost, we expect that the perceived cost after the update is the same as the perceived cost before the update. However, if the perceived cost and the actual cost are of different values, the updated perceived cost would be some place in between the two values. If a linear relationship is presumed, we have

$$\mathbf{C}^{(n)}=\beta\mathbf{c}^{(n-1)}+(1-\beta)\mathbf{C}^{(n-1)}, \tag{7}$$

where parameter $\beta\in[0,1]$ represents the forgetfulness of the travellers, or their activeness in absorbing new information. For the extreme case of $\beta=1$, memory of past knowledge is discarded; on the other hand, $\beta=0$ means that the new information of actual travel cost is not used in the knowledge updating. Rational behaviour would take the value of β in between, i.e. $0<\beta<1$. The bigger value β is, the more travellers rely on new information.

In (7), the updated perceived cost for day $n-1$ is taken as the perceived cost for day n. Travel demand for day n is determined by the demand function (1). Traffic is assigned according to (4) and the actual travel cost is then given by (6). By updating the perceived cost on day n with the actual cost on day n, the perceived cost for day $n+1$ is generated. Now a full circle of travellers' learning process is finished. A dynamical system is formed by combining the equations (1), (4), (6), and (7):

$$\begin{cases} \mathbf{d}^{(n)} = d(\mathbf{C}^{(n)}), \\ \mathbf{f}^{(n)} = diag\{\mathbf{d}^{(n)}\} p(\mathbf{C}^{(n)}), \\ \mathbf{c}^{(n)} = c(\mathbf{f}^{(n)}), \\ \mathbf{C}^{(n+1)} = \beta \mathbf{c}^{(n)} + (1-\beta)\mathbf{C}^{(n)}. \end{cases} \tag{8}$$

Since demand, flow, and actual cost can be subsequently determined as soon as the perceived cost is known, we can simplify the dynamical system (8) to the following form:

$$\mathbf{C}^{(n+1)} = \beta c[diag\{d(\mathbf{C}^{(n)})\} p(\mathbf{C}^{(n)})] + (1-\beta)\mathbf{C}^{(n)}. \tag{9}$$

Here the perceived cost alone is enough to represent the dynamical system. The evolution of the system is characterised by the recurrence function (9) of perceived cost. This recurrence function maps the perceived cost on a day to the perceived cost on the next day.

For the special case of fixed (or inelastic) demand, what matters is not the absolute values of the perceived travel costs but the cost differences between alternative routes on an OD pair (cf. (3)). The system has then only $M-N$ dimensions. Consider now that travel demand is fixed and independent of time (i.e. day n),

$$\mathbf{d} = [\underbrace{d_1, d_1, ..., d_1}_{m_1}, \underbrace{d_2, d_2, ..., d_2}_{m_2}, ..., \underbrace{d_i, d_i, ..., d_i}_{m_i}, ..., \underbrace{d_N, d_N, ..., d_N}_{m_N}]^T. \tag{10}$$

The $M-N$-vector of cost differences can be given in the following form:

$$\begin{aligned} \mathbf{g}_{M-N}{}^{(n)} &= [g_1^{(n)}, g_2^{(n)}, ..., g_{m_1-1}^{(n)}; \\ &\quad g_{m_1}^{(n)}, g_{m_1+1}^{(n)}, ..., g_{m_1+m_2-2}^{(n)}; ...; \\ &\quad g_{M-N-m_N+2}^{(n)}, g_{M-N-m_N+3}^{(n)}, ..., g_{M-N}^{(n)}]^T \\ &= [C_1^{(n)} - C_2^{(n)}, C_1^{(n)} - C_3^{(n)}, ..., C_1^{(n)} - C_{m_1}^{(n)}; \\ &\quad C_{m_1+1}^{(n)} - C_{m_1+2}^{(n)}, C_{m_1+1}^{(n)} - C_{m_1+3}^{(n)}, ..., C_{m_1+1}^{(n)} - C_{m_1+m_2}^{(n)}; ...; \\ &\quad C_{M-m_N+1}^{(n)} - C_{M-m_N+2}^{(n)}, C_{M-m_N+1}^{(n)} - C_{M-m_N+3}^{(n)}, ..., C_{M-m_N+1}^{(n)} - C_M^{(n)}]^T. \end{aligned} \tag{11}$$

Here each OD pair i has $m_i - 1$ entries of cost differences between alternative routes. For the convenience in calculating route choice probabilities, we can transform the above $M-N$-vector into an M-vector by adding 0 entries to it,

$$\begin{aligned} \mathbf{g}_M{}^{(n)} &= [0, g_1^{(n)}, g_2^{(n)}, ..., g_{m_1-1}^{(n)}; \\ &\quad 0, g_{m_1}^{(n)}, g_{m_1+1}^{(n)}, ..., g_{m_1+m_2-2}^{(n)}; ...; \\ &\quad 0, g_{M-N-m_N+2}^{(n)}, g_{M-N-m_N+3}^{(n)}, ..., g_{M-N}^{(n)}]^T. \end{aligned} \tag{12}$$

For simplicity reasons the notation of $\mathbf{g}^{(n)}$ may be used in place of both $\mathbf{g}_{M-N}{}^{(n)}$ and $\mathbf{g}_M{}^{(n)}$ if no confusion is likely to arise. The route choice probability can then be written as

$$\Pr\left\{r,\mathbf{g}^{(n)}\right\}=\frac{1}{\sum_{s\in\mathbf{R}_i}\exp[\theta({g_{M\,s}}^{(n)}-{g_{M\,r}}^{(n)})]},\forall r\in\mathbf{R}_i. \tag{13}$$

Traffic dynamics from day to day can then be represented by the recurrence function of the cost difference vector,

$$\mathbf{g}^{(n+1)}=\beta c_g[diag\{\mathbf{d}^{(n)}\}p(\mathbf{g}^{(n)})]+(1-\beta)\mathbf{g}^{(n)}, \tag{14}$$

where ${\mathbf{c}_g}^{(n)}$ is a transformation of $\mathbf{c}^{(n)}$, in the way similar to (11) or (12).

2.2 The dynamic equilibrium

We consider the dynamical system in the generic form of

$$\mathbf{x}^{(n+1)}=f(\mathbf{x}^{(n)}), \tag{15}$$

where f is the recurrence function and $\mathbf{x}^{(n)}$ represents the state of the dynamical system on day n. For day-to-day traffic dynamics, $\mathbf{x}^{(n)}$ can be $\mathbf{C}^{(n)}$ or $\mathbf{g}^{(n)}$. Here $\mathbf{x}^{(n+1)}$ is the *image* of $\mathbf{x}^{(n)}$ and $\mathbf{x}^{(n)}$ is a *pre-image* of $\mathbf{x}^{(n+1)}$. Once the initial state (or initial point) $\mathbf{x}^{(0)}$ is given, any future state can be derived by iterating the recurrence function,

$$\mathbf{x}^{(n)}\Big|\mathbf{x}^{(0)}=\underbrace{f(f(\ldots f}_{n}(\mathbf{x}^{(0)})\ldots))=f^{(n)}(\mathbf{x}^{(0)}). \tag{16}$$

The notation $\mathbf{x}^{(n)}\Big|\mathbf{x}^{(0)}$ implies that the state on day n ($n=1,2,\ldots$) intrinsically depends on the initial state. The specification of the initial state can be omitted if no confusion is likely to arise. The *trajectory* of the dynamical evolution starting from $\mathbf{x}^{(0)}$ is

$$\begin{aligned}&\mathbf{x}^{(0)},\mathbf{x}^{(1)},\mathbf{x}^{(2)},\ldots,\mathbf{x}^{(n)},\ldots\\&=\mathbf{x}^{(0)},f(\mathbf{x}^{(0)}),f^{(2)}(\mathbf{x}^{(0)}),\ldots,f^{(n)}(\mathbf{x}^{(0)}),\ldots.\end{aligned} \tag{17}$$

If the sequence in (17) has a limit (i.e. the sequence converges), then this limit gives a steady (or stationary) state of the dynamical system. Such a steady state $\mathbf{x}^*$ is the solution of

$$\mathbf{x}^*=f(\mathbf{x}^*), \tag{18}$$

i.e. it is a fixed point of the recurrence function. By repeatedly applying the recurrence function, we have

$$\underbrace{f(f(\ldots f}_{n}(\mathbf{x}^*)\ldots))=\mathbf{x}^*,n=1,2,\ldots \tag{19}$$

This means that a dynamical evolution starting from the fixed point will forever remain at that point (hence ‘stationary’).

Consider the recurrence function of day-to-day traffic dynamics, (9), its fixed point is any perceived cost vector that satisfies

$$\mathbf{C}^*=\beta c[diag\{d(\mathbf{C}^*)\}p(\mathbf{C}^*)]+(1-\beta)\mathbf{C}^*. \tag{20}$$

If β equals 0 then any feasible cost vector will be a fixed point. This is not realistic but it makes sense because $\beta=0$ actually means drivers never update their knowledge (and therefore never bother to change their route choices, resulting in stationary traffic flow). In most realistic cases, β is positive and (20) is equivalent to

$$\mathbf{C}^*=c[diag\{d(\mathbf{C}^*)\}p(\mathbf{C}^*)], \tag{21}$$

which means that the (mean) perceived cost is identical to the actual cost. Therefore a fixed point of (9) is also a stochastic user equilibrium of the static system. The dynamical evolution as represented by (9) signifies the process of achieving equilibrium. The same holds for the case of fixed demand, where $\mathbf{C}^*$ is replaced by $\mathbf{g}^*$.

3 Equilibrium Stability, Attainability and Attraction Domain

The dynamical formulation of equilibrium enables the analysis on equilibrium stability. Stability is important because unstable equilibrium cannot sustain fluctuations and therefore is unlikely to persist in the long run. Stable equilibrium, on the other hand, has an attraction domain which covers a neighbouring area. From this area, all dynamical evolution will converge to the equilibrium. Identifying the exact range of the attraction domain is useful. We can then immediately tell whether a dynamical evolution will converge to the equilibrium by examining the location of the initial state. From any point inside the attraction domain, the evolution will converge to the equilibrium; from any point outside the attraction domain, the evolution will not converge to the equilibrium.

3.1 Attainability, attraction domain and stability

The equilibrium $\mathbf{x}^*$ is *attainable* from an initial point $\mathbf{x}^{(0)}$ if

$$\lim_{n\to\infty} \mathbf{x}^{(n)} \Big| \mathbf{x}^{(0)} = \mathbf{x}^*. \tag{22}$$

We denoted the attainability by $\mathbf{x}^{(0)} \sim \mathbf{x}^*$. The equilibrium is always attainable from itself, i.e. $\mathbf{x}^* \sim \mathbf{x}^*$. If $\mathbf{x}^*$ is attainable from every point in the set $\mathbf{S}$, we can also say that $\mathbf{x}^*$ is attainable from $\mathbf{S}$, denoted as $\mathbf{S} \sim \mathbf{x}^*$. The *attraction domain* (or *attraction basin*) for the equilibrium $\mathbf{x}^*$, denoted as $\mathbf{B}(\mathbf{x}^*)$, is the set of all points that $\mathbf{x}^*$ is attainable from:

$$\mathbf{B}(\mathbf{x}^*) = \{\mathbf{x}^{(0)} : \mathbf{x}^{(0)} \sim \mathbf{x}^*\}. \tag{23}$$

Obviously, $\mathbf{x}^* \in \mathbf{B}(\mathbf{x}^*)$ and $\mathbf{B}(\mathbf{x}^*) \sim \mathbf{x}^*$.

The equilibrium $\mathbf{x}^*$ is *(asymptotically) stable* if its attraction domain contains at least a local neighbourhood, i.e.

$$\exists \delta > 0 : \left\| \mathbf{x}^{(0)} - \mathbf{x}^* \right\| < \delta \Rightarrow \lim_{n\to+\infty} \mathbf{x}^{(n)} = \mathbf{x}^*. \tag{24}$$

For an unstable equilibrium, no matter how close to the equilibrium the dynamical evolution has started, there is no guarantee that the evolution will converge to the equilibrium.

3.2 Characterisation of the attraction domain

To identify the exact range of the attraction domain, we first present some topological characterisations.

Theorem 1 Suppose that $\mathbf{x}^* = f(\mathbf{x}^*)$ is an equilibrium point of the dynamical system $\mathbf{x}^{(n+1)} = f(\mathbf{x}^{(n)})$, then $\mathbf{B}(\mathbf{x}^*)$ is strictly invariant. Moreover, if f is continuous and $\mathbf{x}^*$ is stable, then $\mathbf{B}(\mathbf{x}^*)$ is open and its boundary, if non-empty, is invariant and formed by trajectories.

Proof: See [4]. □

The invariant property means that a trajectory lies either entirely inside the attraction domain or entirely outside. That is, if $f^{(n)}(\mathbf{x}^{(0)}) \in \mathbf{B}(\mathbf{x}^*)$ for some $n = 0,1,2,\ldots$, then

$f^{(n)}(\mathbf{x}^{(0)}) \in \mathbf{B}(\mathbf{x}^*)$ for all n. The openness means that points on the boundary are not attracted to the equilibrium. Furthermore, the trajectory from a point on the boundary will forever remain on the boundary.

Theorem 2 Consider the dynamical system $\mathbf{x}^{(n+1)} = f(\mathbf{x}^{(n)})$. Suppose that every trajectory has a limit, i.e. every point in the state space converges to equilibrium (one or another, stable or unstable). Denote $\{\mathbf{x}^*\}$ as the set of equilibrium points, then

$$\mathbf{B}(\mathbf{x}_i^*) \cap \mathbf{B}(\mathbf{x}_j^*) = \mathbf{\Phi}, \forall \mathbf{x}_i^*, \mathbf{x}_j^* \in \{\mathbf{x}^*\}, \mathbf{x}_i^* \neq \mathbf{x}_j^*, \tag{25}$$

Moreover, if f is continuous, then the boundary of a stable equilibrium's attraction domain is formed by trajectories to unstable equilibrium(s).

Proof: The mutual exclusiveness in (25) follows directly from the definition of attraction domain. A point cannot converge to more than one equilibrium point and therefore must be in one attraction domain and not be in any other attraction domain.

From Theorem 1 we know that the boundary of the stable equilibrium $\mathbf{x}^*$ is formed by trajectories. In the following, we will prove that each of these trajectories must end at an unstable equilibrium. We prove this by contradiction. Because every trajectory has a limit, it converges either to a stable equilibrium or to an unstable equilibrium. Suppose that there is one trajectory on the boundary which converges to another stable equilibrium $\mathbf{x}'^*$. Then any point $\mathbf{z}$ on this trajectory belongs to the open set $\mathbf{B}(\mathbf{x}'^*)$. Therefore, by the definition of open set, there must exist a neighbourhood of $\mathbf{z}$ that lies entirely inside $\mathbf{B}(\mathbf{x}'^*)$. On the other hand, because $\mathbf{z} \in \partial\mathbf{B}(\mathbf{x}^*)$, any neighbourhood of $\mathbf{z}$ must overlap with $\mathbf{B}(\mathbf{x}^*)$. Therefore some points in this neighbourhood belongs to both $\mathbf{B}(\mathbf{x}'^*)$ and $\mathbf{B}(\mathbf{x}^*)$. This contradicts the fact that the domains of attraction for two different equilibria cannot overlap and therefore their intersection must be empty. Hence, no trajectory on the boundary of $\mathbf{B}(\mathbf{x}^*)$ converges to a stable equilibrium. □

This theorem provides an important outlook of the state space. The state space can be partitioned into mutually exclusive subsets. Each subset is associated with an equilibrium point and represents the equilibrium's attraction domain. If such a partition chart has been drawn, given an initial point we can immediately tell the eventual equilibrium of the dynamical evolution without knowing the exact form of the recurrence function.

Moreover, the attraction domain of the stable equilibrium can be drawn by focusing on the unstable equilibria. When we have identified all the trajectories to unstable equilibria, the boundary of the stable equilibrium's attraction domain is also formed. We show this by the following example.

Example 1 Consider a network [3] with one OD pair connected by three routes. Demand is fixed at 2 units. The cost functions for the three routes are given as:

$$c_1(\mathbf{f}) = f_1 + 3f_2 + 1, c_2(\mathbf{f}) = 2f_1 + f_2 + 2, c_3(\mathbf{f}) = f_3 + 6.$$

The dispersion parameter is $\theta = 1$ and the updating ratio is $\beta = 0.2$. There are multiple equilibria. All system evolution converges to one of the following three equilibria:

$$\mathbf{x}_I^* = [1.752, 0.151, 0.097]^{\mathrm{T}}; \mathbf{x}_{II}^* = [0.768, 1.031, 0.201]^{\mathrm{T}}; \mathbf{x}_{III}^* = [0.226, 1.588, 0.186]^{\mathrm{T}},$$

or, in cost differences, given as

$$(g_1, g_2)_I^* = (-2.45, -2.89); (g_1, g_2)_{II}^* = (0.29, -1.34); (g_1, g_2)_{III}^* = (1.95, -0.20).$$

The actual attraction domains for the three equilibria can be identified by examining the phase portrait in Figure 1. The phase portrait is drawn by tracing the evolution from numerous points in the state space. We can see from the phase portrait that all points on the left side of dashed curve are attracted to the stable equilibrium $\mathbf{g}_I^*$; all points right to the stable equilibrium $\mathbf{g}_{III}^*$. The points on the dashed curve actually give the attraction domain of $\mathbf{g}_{II}^*$, an unstable equilibrium.

The exact range of the attraction domains can also be determined without the burdensome task of drawing the phase portrait. Instead of tracing the trajectory from each point in the state space, we focus on the trajectories towards the unstable equilibrium. We first identify the trajectories towards $\mathbf{g}_{II}^*$. This can be done by tracing back from the unstable equilibrium (applying the inverse recurrence function). The result is shown as the dashed curve in Figure 1. The curve defines the boundary of both $\mathbf{B}(\mathbf{g}_I^*)$ and $\mathbf{B}(\mathbf{g}_{III}^*)$. Therefore the set of points left to the curve gives $\mathbf{B}(\mathbf{g}_I^*)$ and the set of points right gives $\mathbf{B}(\mathbf{g}_{III}^*)$. ◊

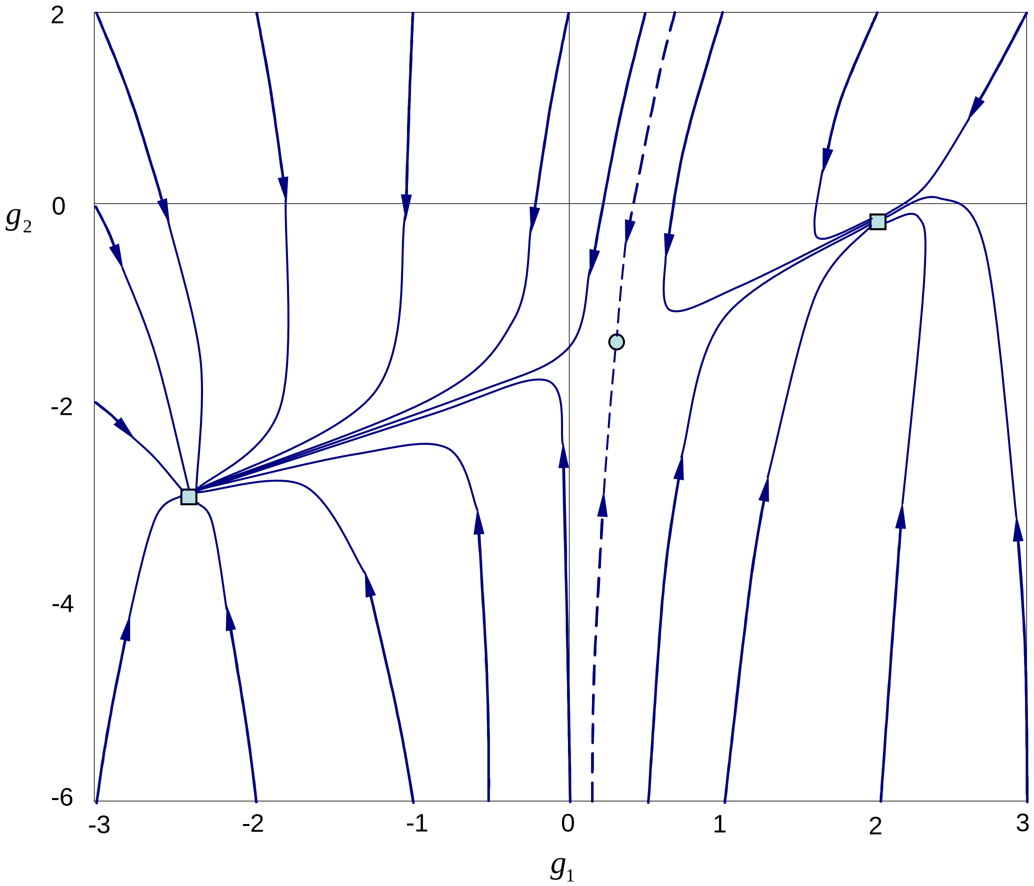

Figure 1: *Phase portrait and attraction domains*

Tracing back from unstable equilibrium can be cumbersome tasks. This is especially true when the dimension of the system is very high and therefore the dimension of an unstable equilibrium's attraction domain is also high. The method, while producing sound theoretical results, may be impractical in real world implementations. It is therefore necessary to look for method that is feasible even at high dimensions.

3.3 Estimation of attraction domain

Two methods for estimating the attraction domain are illustrated here. They are handy when determining the exact range of attraction domains is difficulty or not necessary. The first method utilises the Lyapunov function. For the theoretical background of this method, see [4]. The second method is a sampling method. It applies to network of any dimension but it might bring in errors in the estimation results, which can be reduced but not eliminated. We show these two methods through examples.

Example 2 Consider the same network as in Example 1. Now we adopt the Lyapunov function for the two stable equilibria $\mathbf{g}_I^*$ and $\mathbf{g}_{III}^*$ and obtain the estimates of their attraction domains $\mathbf{E}(\mathbf{g}_I^*) \subset \mathbf{B}(\mathbf{g}_I^*)$ and $\mathbf{E}(\mathbf{g}_{III}^*) \subset \mathbf{B}(\mathbf{g}_{III}^*)$. First we consider the equilibrium $\mathbf{g}_I^*$ and choose its Lyapunov function as the Euclidean distance square, $V_I(\mathbf{g}) = (\mathbf{g} - \mathbf{g}_I^*)^T(\mathbf{g} - \mathbf{g}_I^*)$. We then maximise the domain of this function while keeping it as an invariant set and the Lyapunov conditions satisfied. The estimate result of $\mathbf{B}(\mathbf{g}_I^*)$ is given as

$$\mathbf{E}(\mathbf{g}_I^*) = \{\mathbf{x} : V_I(\mathbf{x}) < 4.444\},$$

the boundary of which is shown as the solid circle in Figure 2. All points inside this circle converges to $\mathbf{g}_I^*$.

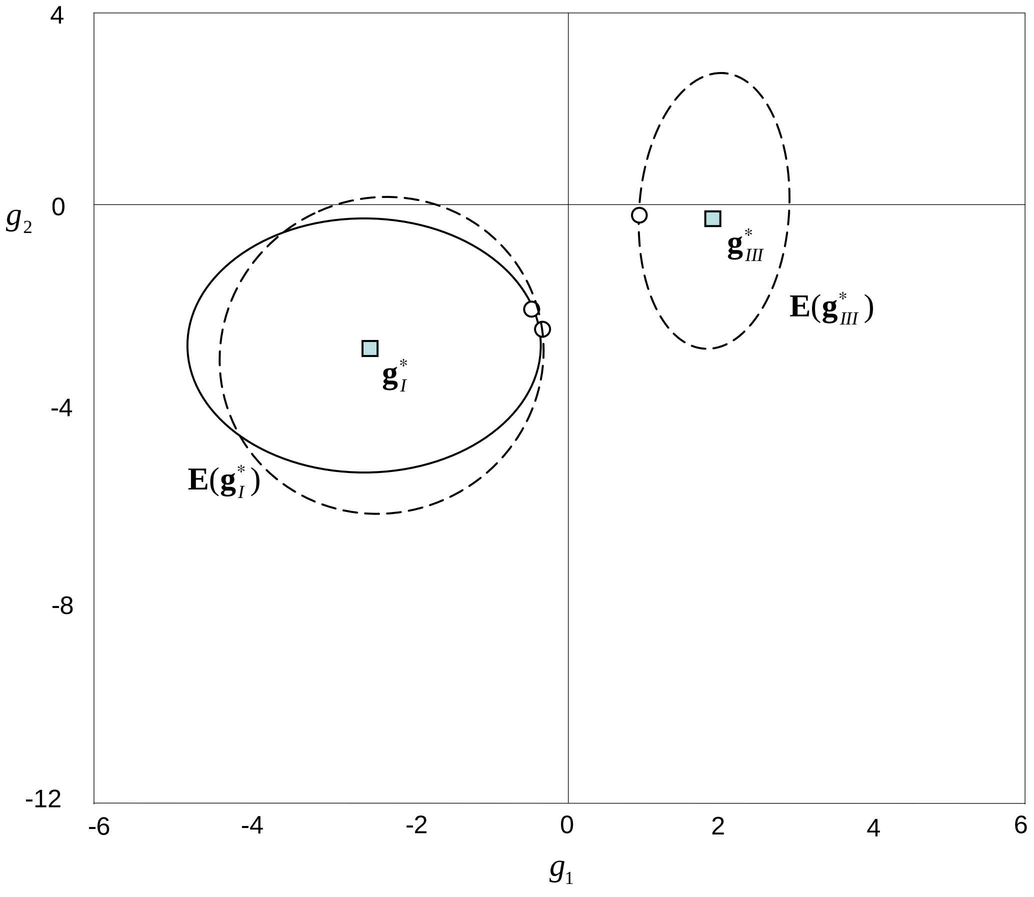

Figure 2: *Attraction domain estimation by Lyapunov function*

If a different Lyapunov function is used, the estimation result is also different. If we use the Lyapunov function $V_I'(\mathbf{g}) = (\mathbf{g} - \mathbf{g}_I^*)^T \mathbf{P}(\mathbf{g} - \mathbf{g}_I^*)$, where

$$\mathbf{P} = \begin{bmatrix} 4.795 & 0.508 \\ 0.508 & 2.396 \end{bmatrix},$$

then the estimate of $\mathbf{B}(\mathbf{g}_I^*)$ is given as

$$\mathbf{E}(\mathbf{g}_I^*) = \{\mathbf{x} : V_I'(\mathbf{x}) < 20.949\}.$$

Its boundary is shown by the dashed line around $\mathbf{g}_I^*$ in Figure 2.

Similarly for $\mathbf{g}_{III}^*$ we take $V_{III}(\mathbf{g}) = (\mathbf{g} - \mathbf{g}_I^*)^T \mathbf{P}(\mathbf{g} - \mathbf{g}_I^*)$ with

$$\mathbf{P} = \begin{bmatrix} 11.150 & -0.927 \\ -0.927 & 2.017 \end{bmatrix}.$$

The estimate of $\mathbf{B}(\mathbf{g}_{III}^*)$ is given as

$$\mathbf{E}(\mathbf{g}_{III}^*) = \{\mathbf{x} : V_{III}(\mathbf{x}) < 13.391\}.$$

Its boundary is shown by the dashed line around $\mathbf{g}_{III}^*$ in Figure 2. ◊

We can see from the above example that the selection of the Lyapunov function (or the selection of $\mathbf{P}$ in particular) has influence on the estimated range of attraction domain. A different Lyapunov function may produce an estimate much larger in range. It would therefore be useful if we could find the 'optimal' Lyapunov function. We also see that the estimation by Lyapunov function is at times rather conservative. This is especially so when the recurrence functions are highly irregular (e.g. nonlinear, non-monotone). The second method applies to recurrence function of any form and does not require *a priori* knowledge of equilibrium solution.

Example 3 Consider the same network as in Example 1. We now perform the sampling method to estimate the attraction domains. To do so, we first select a set of initial points, given by the integer points in the following set:

$$\{-3 < g_1 < 3; -6 < g_2 < 2\}.$$

There are totally 35 points, as shown in Figure 3. We then trace their trajectories and observe the destiny of their evolution over time. It takes less than 100 steps to see that all the 21 initial points on the left (represented by cross) converge to the equilibrium $\mathbf{g}_I^*$, while all the 14 initial points on the right converge to the equilibrium $\mathbf{g}_{III}^*$.

We can then draw the attraction domain estimation chart as in Figure 4. The estimation is made by connecting the points that are furthest from the equilibrium and these line segments form the boundary of the estimate. The boundary for $\mathbf{E}(\mathbf{g}_I^*)$ is $\{g_1 = 0; -6 < g_2 < 2\}$ and the boundary for $\mathbf{E}(\mathbf{g}_{III}^*)$ is $\{g_1 = 1; -6 < g_2 < 2\}$.

The range left out, $\{0 < g_1 < 1; -6 < g_2 < 2\}$, can be gradually reduced by selecting more points in the range and performing a second round of simulation. We choose as our simulation points the 28 points that are evenly distributed in the above range, as shown in Figure 5. We can trace their evolution to the two stable equilibrium points. Again, it takes less than 100 steps before convergence. The resulting attraction domain estimation is shown in Figure 6, which is an expansion of the estimation in Figure 4. ◊

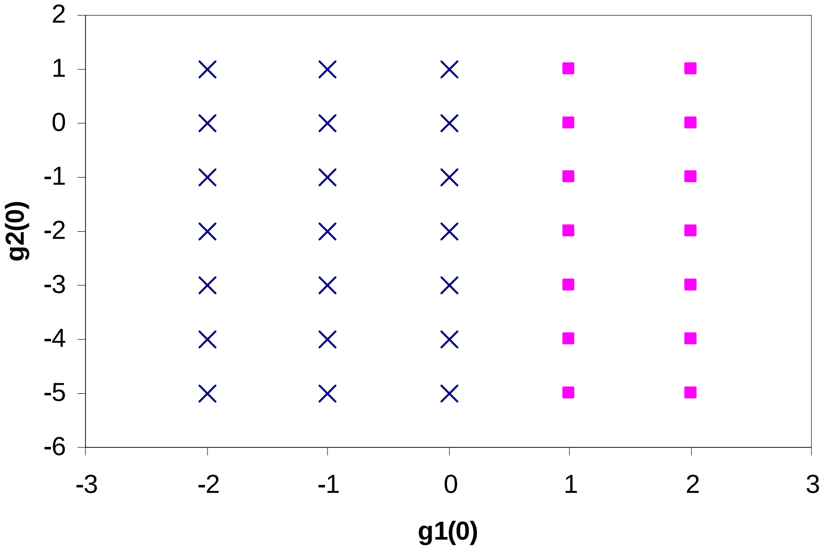

Figure 3: *Simulation points on day 0*

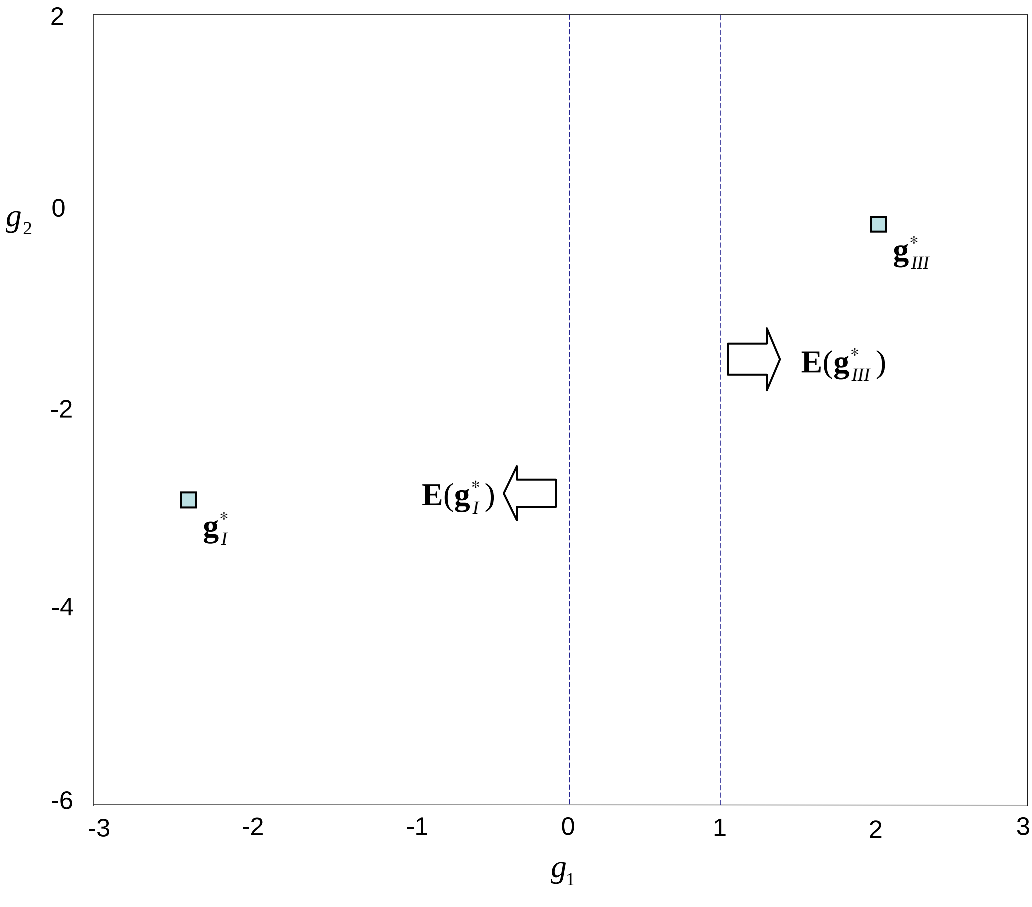

Figure 4: *Attraction domain estimation by simulation*

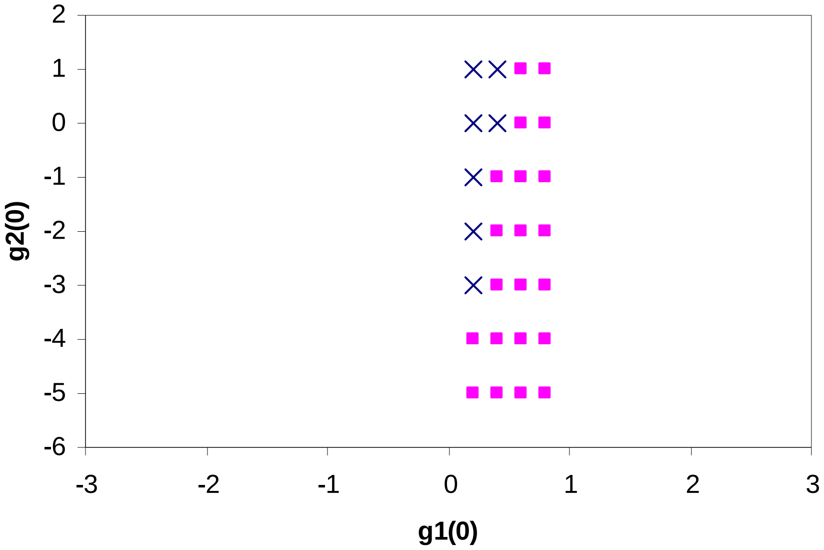

Figure 5: *Closer simulation points on day 0*

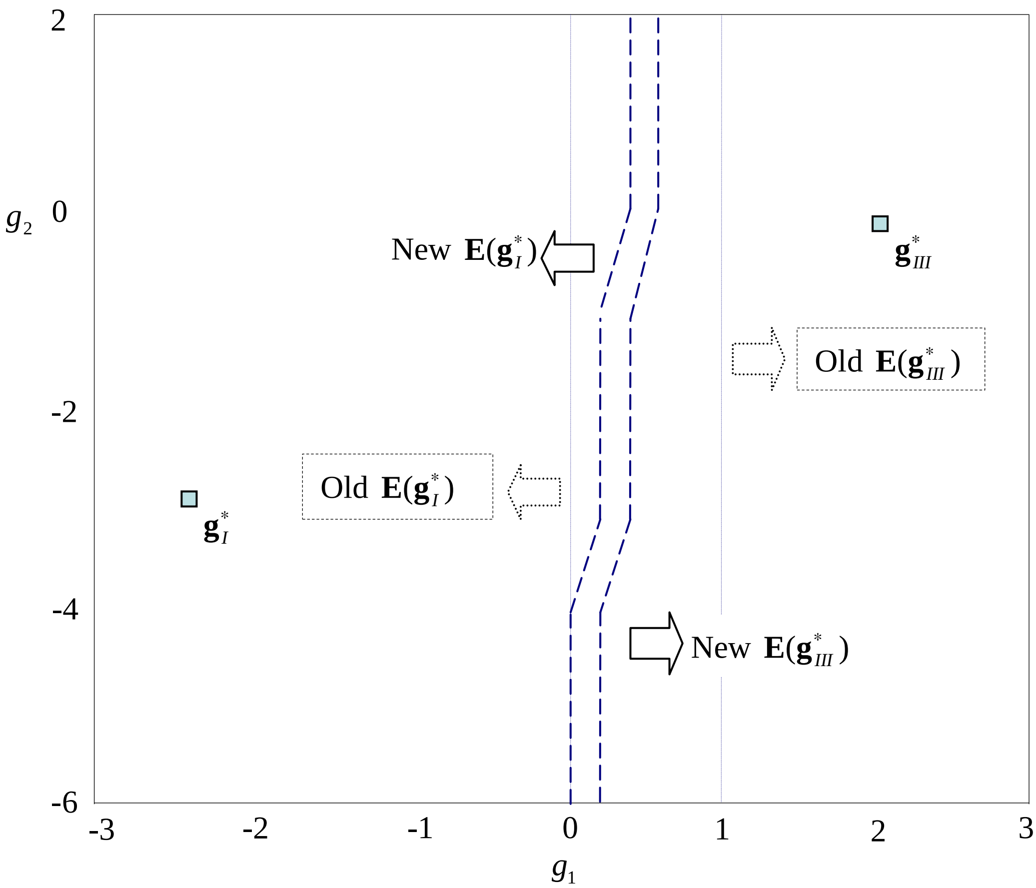

Figure 6: *Expanding the attraction domain estimation by simulation*

This method can automatically obtain the stable equilibrium solutions. That is, even if we do not know the equilibrium solutions before performing the simulation, we can obtain stable equilibrium solutions after a few steps of iterations. However, unstable equilibrium cannot be obtained in this way. We also notice through Figure 5 and Figure 6 that the estimates can always be refined by another round of simulation for the points in the left-out range. The estimated attraction domain can then be expanded in the direction of this left-out range. As the direction-based search keeps on, we would expect that the left-out range shrinks and converges to a curve identical to the dashed curve in Figure 1. This curve presents the boundary of the two attraction domains.

Here we have assumed that the attraction domain is connected and convex. A convex set is one such that for any two points in the set, the line segment between the two points lies wholly in the set. If we know a number of points in the attraction domain, the line segment connecting any two of the points also belongs to the attraction domain. The outmost segments then form the boundary of the estimated attraction domain, as shown in the above example. However, the attraction domain is not always convex, such as $\mathbf{B}(\mathbf{g}_I^*)$ shown in Figure 1 which is concave. So there is some possible error in the estimate where some non-convergent points may have been included. This type of error can be reduced by improving the precision of simulation (i.e. denser simulation points) but cannot be completely eliminated as long as the attraction domain is not convex.

4 Simultaneous Route and Departure Time Choice

The same dynamical system formulation for day-to-day traffic dynamics can also be applied to simultaneous route and departure time choice (SRD) problems. Besides routes, travellers also choose a departure time for day-to-day. We presume that no penalty applies for early or late arrivals. Instead, travellers choose the route and departure time with minimal cost.

4.1 Formulation of day-to-day SRD

The departure time span $[0,T]$ is discretised into a total of k intervals, $[0,T/k),[T/k,2T/k),...,[(k-1)T/k,T]$. Choices of departure time within the same interval are considered identical to each other in terms of preference and actual travel cost. On day n, travellers' knowledge of the network is represented by the $M\times k$ vector of mean perceived route costs at each departure intervals,

$$\mathbf{C}^{(n)}=[C_{1,1}^{(n)},C_{1,2}^{(n)},...,C_{1,k}^{(n)};C_{2,1}^{(n)},C_{2,2}^{(n)},...,C_{2,k}^{(n)};...;C_{M,1}^{(n)},C_{M,2}^{(n)},...,C_{M,k}^{(n)}]^{\mathrm{T}}. \tag{26}$$

The probability of choosing a route and departure time pair (r,t), $r=1,2,...,M;t=1,2,...,k$, under the given cost $\mathbf{C}^{(n)}$ is determined by

$$\Pr\{r,t;\mathbf{C}^{(n)}\}=\frac{1}{1+\sum_{s\in\mathbf{R}_i,s\neq r;\tau=1,2,...,k,\tau\neq t}\exp[\theta(C_{r,t}^{(n)}-C_{s,\tau}^{(n)})]}. \tag{27}$$

The actual traffic costs are then determined by the travel cost (or performance) functions:

$$\begin{aligned}\mathbf{c}^{(n)}=[&c_{1,1}(\mathbf{f}^{(n)}),c_{1,2}(\mathbf{f}^{(n)}),...,c_{1,k}(\mathbf{f}^{(n)});c_{2,1}(\mathbf{f}^{(n)}),c_{2,2}(\mathbf{f}^{(n)}),...,c_{2,k}(\mathbf{f}^{(n)});\\&...;c_{M,1}(\mathbf{f}^{(n)}),c_{M,2}(\mathbf{f}^{(n)}),...,c_{M,k}(\mathbf{f}^{(n)})]^{\mathrm{T}}.\end{aligned} \tag{28}$$

Travellers' perceived cost is updated daily in a way similar to (7).

We can see here that departure time choices can be treated similarly as route choices, only that they bring a higher dimension (or degree of freedom) to the system and make the cost functions more complicated, as in (28). Except the higher dimensions, the structure of the dynamical system remains the same.

4.2 Stability of SRD equilibrium

Because of higher dimensions, stability of SRD equilibrium is more difficult to attain. This follows the observation that the perturbations can then come from more dimensions. As in (24), stability requires the capability of containing perturbations from any dimension or combination of dimensions.

The higher dimension in SRD problem may also increase the probability of non-equilibrium attractors, namely, cycles and chaos. A cycle is a periodic attractor which circulates among a set of points in a specific order. A chaotic attractor, on the other hand, exhibits no periodicity. Real world observations of non-stationary traffic flow over time may suggest the existence of such attractors.

5 Conclusions

In this paper we have formulated the day-to-day route (and departure time) choice problem as a dynamical system. Perceived cost is updated every day and the resulting route choices also change from day to day. The dynamic equilibrium is achieved when a stationary choice pattern is formed in the system. Stability of an equilibrium solution depends on the shape of its attraction domain. If the attraction domain is fully developed in dimension then the equilibrium is stable. This means that the system allows any perturbations on any dimension without diverging from the equilibrium. The range of the attraction domain, containing all the points that will dynamically evolve to the equilibrium, characterised the attractiveness of the equilibrium as well as the attainability of the equilibrium when an initial point is predefined.

We have shown that the attraction domain for a stable equilibrium is always open, with its boundary formed by trajectories towards unstable equilibria. Therefore the exact range of the attraction domain can de determined by tracing back from the unstable equilibria. Estimate of the attraction domain can also be obtained by utilising the Lyapunov function or performing one or several rounds of simulation.

The simultaneous route and departure time choice problem in the day-to-day setting can be similarly formulated as that of route choice alone. The only difference is that departure time choices bring more degrees of freedom and therefore makes the system more complicated. Stability of SRD equilibrium is generally more difficult to attain because of higher dimensions.

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