## Summary

This thesis contains a number of new contributions regarding Turán problems for hypergraphs. These new results involve the Turán number of Berge linear forests, Turán numbers of star forests (in three different variants), linear Turán numbers of generalized crowns, and connected Turán numbers for Berge paths.

Starting with Turán problems for (ordinary) graphs, the most general

known result is the Erdős-Stone-Simonovits Theorem, which states that the maximum number of edges in an $n$-vertex graph avoiding the forbidden subgraph $H$ is asymptotically the $1-\frac{1}{r}$ fraction of the number of edges of the complete graph, where $r+1$ is the chromatic number of $H$. This Erdős-Stone-Simonovits Theorem gives an asymptotically tight result whenever $H$ is not bipartite. However, when $H$ is a bipartite graph, the Erdős-Stone-Simonovits Theorem just implies that the Turán number of $H$, denoted by $ex(n,H)$, is $o(n^{2})$. This provides very little information about the exact value of $ex(n,H)$. Hence, it is interesting to study the Turán numbers of bipartite graphs, and characterize the extremal graphs.

 A classic result due to Erdős and Gallai about paths states that the number of edges $e(G)$ of an $n$-vertex graph $G$ containing no copy of a path $P\_{k+1}$ of length $k$ as its subgraph satisfies: $e(G)\leq \frac{(k-1)n}{2}$, with equality holding if and only if $k$ divides $n$ and $G$ consists of vertex-disjoint copies of the complete graph $K\_{k}$ on $k$ vertices. It is also known that an $n$-vertex graph $G$ containing no copy of a star $S\_{k}$ on $k+1$ vertices as its subgraph satisfies: $e(G)\leq \frac{(k-1)n}{2}$, with equality holding if and only if $G$ is a $(k-1)$-regular graph. In 1962, Erdős and Sós conjectured that the above results about paths and stars also hold for any tree of the same size, that is, every graph on $n(\geq k+1)$ vertices having at least $\frac{(k-1)n}{2}+1$ edges must contain every tree on $k+1$ vertices.

 It is natural to consider determining the Turán number of forests, since a forest consists of one or more trees. In this context, another classic

result of Erdős and Gallai involving matchings states that $ex(n,M\_{k})=max\{\left(\begin{array}{c}k-1\\2\end{array}\right)+(k-1)(n-k+1),\left(\begin{array}{c}2k-1\\2\end{array}\right)\}$, where $M\_{k}$ denotes a matching consisting of $k$ independent edges. Indeed, a matching can be viewed as a linear forest (a forest consisting of paths) or star forest ( a forest consisting of stars). For a general forest, Brandt proved that any $n$-vertex graph containing more than $max\{\left(\begin{array}{c}k-1\\2\end{array}\right)+(k-1)(n-k+1),\left(\begin{array}{c}2k-1\\2\end{array}\right)\}$ edges contains every forest on $k$ edges without isolated vertices. Following up on this, Lidický, Liu and Palmer considered linear forests and star forests, and determined their Turán numbers asymptotically (for sufficiently large $n$).

 Motivated by the above results, in this thesis we focus on the Turán numbers of linear forests and star forests in hypergraphs. For an integer $r\geq 2$, an $r$-uniform hypergraph consists of a vertex set $V$ and a set of hyperedges, each of which is a subset of $V$ with $r$ vertices. There exist various ways to generalize the relevant notions from graphs to $r$-uniform hypergraphs. In this thesis, we consider three different hypergraph settings that are known from literature. Fix a graph $F$ and an integer $r\geq 2$. Then the expansion of $F$is the $r$-uniform hypergraph $F^{+}$ constructed by adding $r-2$ new distinct vertices to each edge of $F$, where all added new vertices are distinct. When $r=2$, the expansion $F^{+}$ is just the original graph $F$. An $r$-uniform hypergraph is called linear if every pair of vertices is contained in at most one hyperedge. It is easy to see that each expansion is a linear hypergraph. However, not every linear hypergraph is an expansion of some graph. Indeed, if a linear hypergraph contains a hyperedge $e$ each vertex of which is contained in another hyperedge distinct from $e$, then it cannot be an expansion of any graph. Let $F$ be a graph. An $r$-uniform hypergraph $H$ is called a Berge-$F$ if there exist an injection $f:V(F)\rightarrow V(H)$and a bijection $f^{'}:E(F)\rightarrow E(H)$such that for every edge $uv\in E(F)$ we have $\{f(u),f(v)\}⊆f^{'}(uv)$. For a given graph $F$, we can observe that there are many different hypergraphs that are Berge-$F$, and that a fixed hypergraph $H$ can be a Berge-$F$ for more than one graph $F$.

 Gerbner and Palmer obtained the following inequality about the Turán number for a Berge-$F$ and the generalized Turán number for the graph $F$:$ex(n,K\_{r},F)\leq ex\_{r}(n,Berge-F)\leq ex(n,K\_{r},F)+ex(n,F)$.

Here $ex(n,K\_{r},F)$ denotes the maximum number of distinct copies of $K\_{r}$ in an $n$-vertex $F$-free graph.

Denote by $N(H,G)$ the number of distinct copies of the graph $H$ in the graph $G$. Given a graph $G$, by using two colors red or blue to color the edges of $G$, we obtain a red-blue graph. Denote by $G\_{red}$ and $G\_{blue}$ the subgraphs of $G$ consisting of (only) the red edges and (only) the blue edges, respectively. For a fixed red-blue graph $G$ and a positive integer $r$, we define $g\_{r}(G)=e(G\_{red})+N(K\_{r},G\_{blue})$. Gerbner, Methuku and Palmer obtained the following result:$ex\_{r}(n,Berge-F)\leq max\{g\_{r}(G):G is an n-vertex F-free red-blue graph\}$.For matchings, Kang, Ni and Shan obtained the Turán number $ex\_{r}(n,Berge-M\_{k+1})$ when $r\leq k-1$ and $r\geq 2k+2$.

Turning to a given family of graphs $F$ instead of one graph $F$, we also have that $ex(n,K\_{r},F)\leq ex\_{r}(n,Berge-F)\leq ex(n,K\_{r},F)+ex(n,F)$. Moreover, $ex\_{r}(n,Berge-F)\leq max\{g\_{r}(G):G is an n-vertex F-free red-blue graph\}$. In Chapter 2, by using a closure operation on red-blue graphs, we obtain the Turán number $ex\_{r}(n,Berge-L\_{n,k})$, where $L\_{n,k}$ is a family of linear forests with $n$ vertices and $k$ edges.

Erdős determined the Turán number of the expansion of $M\_{k}$, when the number of vertices is sufficiently large. Later, Khormali and Palmer generalized Erdős' result to the expansion $k∙S\_{l}^{+}$ of a star forest $k∙S\_{l}$ composed of $k$ copies of the star $S\_{l}$. In Chapter 3, we generalize the above result by obtaining the Turán number of the expansion of a general star forest. A seminal result involving linear Turán numbers settles the problem of determining the linear Turán number $ex\_{3}^{lin}(n,C\_{3}^{+})$ of the expansion of $C\_{3}$, which is equivalent to the famous (6,3)-problem. Motivated by this result, more and more papers on this topic have emerged. In 2022, Khormali and Palmer determined the linear Turán number of $k∙S\_{l}^{+}$ asymptotically (for sufficiently large $n$). In Chapter 3, we determine the linear Turán number of a general hypergraph star forest asymptotically. Khormali and Palmer also determined the Turán number of a Berge-$k∙S\_{l}$ asymptotically. With our final two main results in Chapter 3, we extend the above results to more general Berge-star forests.

In the context of hypergraphs, the crown is a linear 3-graph which is obtained from three pairwise disjoint hyperedges by adding one hyperedge that intersects all three of them. As an important research object, Gyárfás, Ruszinkó and Sárközy initiated the study on the linear Turán number of the crown. Recently, in a paper of 2022, Tang, Wu, Zhang and Zheng obtained a sharp upper bound for it. In Chapter 4, we give two natural generalizations of the crown to linear $r$-uniform hypergraphs, and obtain an upper bound and lower bound for the linear Turán number.

Recall from earlier in this summary that Erdős and Gallai's result about paths implies that the extremal graphs for the Turán numbers of paths are disconnected. The same holds for the extremal hypergraphs for the Turán numbers of Berge paths. In Chapter 5, we focus on connected Turán numbers of Berge paths. Apart from several other results, in our main result we obtain a sharp asymptotic upper bound for the connected Turán number of Berge paths $ex\_{r}^{conn}(n,Berge-P\_{k+1})$when $r\geq k$.

Although this thesis contains a number of new contributions to the field of Turán numbers for hypergraphs, there are some interesting problems and conjectures that remain unresolved. We also posed several new open problems and conjectures. We hope that these will attract the attention of many researchers and spur the developments in this area.