Multi-class Queues and Stochastic Networks

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Multi-class Queues and Stochastic Networks

Detailed content:

1. reversibility, stationarity, basic queues, output theorem, feedforward networks
2. partial balance, Jackson network, Kelly-Whittle netwerk, arrival theorem
3. quasi-reversibility, customer types, BCMP networks, bandwidth sharing networks
4. blocking, aggregation, decomposition
5. loss networks, insensitivity via supplementary variables
6. sojourn time distribution in networks
7. MVA, AMVA, QNA
8. fluid queues, basic models
9. feedback fluid queues, networks of fluid queues

Multi-class Queues and Stochastic Networks
Today (lecture 6): MVA, AMVA, QNA

Handbook, chapter 13 – published handbook on website MQSN

- MVA for single queues
- MVA for closed networks

Preliminaries

- Product form equilibirum distribution not enough for performance analysis:

closed network : equilibrium distribution

- Theorem: The equilibrium distribution for the closed Jackson network containing N jobs is

$$\pi(n) = B_N \prod_{j=1}^{J} \rho_j^{n_j} \quad \rho_j = \gamma_j / \mu_j \quad n \in S_N = \{n : \sum_j n_j = N\}$$

and satisfies partial balance

$$\sum_{k=1}^{J} \pi(n) q(n, n - e_j + e_k) = \sum_{k=1}^{J} \pi(n - e_j + e_k) q(n - e_j + e_k, n)$$

traffic equations $\sum_k \gamma_j p_{jk} = \sum_k \gamma_k p_{kj}$

$$q(n, n - e_j + e_k) = \mu_j p_{jk}$$

$$q(n, n - e_j) = \mu_j p_{j0}$$

$$q(n, n + e_k) = \lambda_k$$

Open and closed networks

$$\lambda_i = \gamma_i + \sum_{j=1}^{N} \lambda_j \cdot r_{ji} \qquad i = 1,\ldots,N$$

- Theorem
  - steady state distribution (if open: provided $\rho_i = \lambda_i / \mu_i < 1$):

$$P(k_1,\ldots,k_N) = C \prod_{i=1}^{N} \rho_i^{k_i}$$

Intermezzo: Normalising constant for open networks

- Determining the normalising constant easy for open networks

$$P(k_1,\ldots,k_N) = C\prod_{i=1}^{N}\rho_i^{\,k_i} = \prod_{i=1}^{N}(1-\rho_i)\cdot\rho_i^{\,k_i}$$

$$C^{-1} = \sum_{(k_1,\ldots,k_N):k_i\geq 0}\ \prod_{i=1}^{N}\rho_i^{\,k_i} = \prod_{i=1}^{N}\sum_{k_i\geq 0}\rho_i^{\,k_i} = \prod_{i=1}^{N}(1-\rho_i)^{-1}$$

Determining the normalising constant: Buzen's algorithm

- Recursion for summation

$$C_m^{-1} = \sum_{k_1+\ldots+k_N=m} \prod_{i=1}^{N} \rho_i^{k_i}$$

Define

$$G_i(k), \quad i = 0,1,2,\ldots,N \quad k = 0,\ldots,m$$

with initial values

$$G_1(k) = \rho_1^{k} \qquad k = 0,\ldots,m$$

$$G_i(0) = 1 \qquad i = 0,1,2,\ldots,N$$

recursion

$$G_i(k) = G_{i-1}(k) + \rho_i G_i(k-1)$$

then

$$C_m^{-1} = G_N(m)$$

Correctness of Buzen’s algorithm

$$C_m^{-1} = \sum_{k_1+\ldots+k_N=m} \prod_{i=1}^{N} \rho_i^{k_i}$$

$$\sum_{k_1+k_2=m} \prod_{i=1}^{2} \rho_i^{k_i} = \sum_{k_1+k_2=m} \rho_1^{k_1} \rho_2^{k_2} = \sum_{k_1=0}^{m} \rho_1^{k_1} \rho_2^{m-k_1}$$

$$= \rho_1^{m} + \sum_{k_1=0}^{m-1} \rho_1^{k_1} \rho_2^{m-k_1} = \rho_1^{m} + \rho_2 \sum_{k_1=0}^{m-1} \rho_1^{k_1} \rho_2^{m-1-k_1}$$

$$G_1(k) = \rho_1^{k} \qquad k = 0,\ldots,m$$

$$G_i(k) = G_{i-1}(k) + \rho_i G_i(k-1)$$

Preliminaries

- Product form equilibirum distribution not enough for performance analysis
- But what about the normalising constant?

- Use algorithms that focus on the mean performance measures

- Little's formula $L = \lambda S, \ \ Q = \lambda W \quad \rho = \lambda b.$

- PASTA, MUSTA (ASTA)

Arrival theorem

- PASTA:
  The distribution of the number of customers in the system seen by a a customer arriving to a system according to a Poisson process (i.e., at an arrival epoch) equals the distribution of the number of customers at an arbitrary epoch.

- Arrival theorem (open Jackson network):
  In an open network in equilibrium, a customer arriving to queue j observes the equilibrium distribution.

- Arrival theorem (closed Jackson network):
  In a closed networkin equilibrium, a customer arriving to queue j observes the equilibrium distribution of the network containing one customer less.

MVA for single-station systems: MM1

$$W = \rho \frac{1}{\mu} + Q \frac{1}{\mu}$$

Service upon arrival: PASTA

$$Q = \lambda W$$

$$W = \frac{\rho}{1-\rho} \frac{1}{\mu}$$

MVA for single-station systems: MG1

$$W = \rho R + Qb$$

Residual service time upon arrival:
PASTA

$$Q = \lambda W$$

$$R = \frac{b^{(2)}}{2b} = \frac{b}{2}(1 + c^2) \qquad c^2 = (b^{(2)} - b^2)/b^2$$

$$W = \frac{\rho R}{1 - \rho}$$

MVA for single-station systems: MG1 w priorities

$$W_i = \frac{\sum_j \rho_j R_j}{(1-\sum_{j\le i}\rho_j)(1-\sum_{j<i}\rho_j)}, \qquad i = 1,2,\ldots,C.$$

Arrival theorem

- PASTA:
  If customers arrive according to a Poisson process, then the probability distribution of the number of customers at an arrival time is equal to the probability distribution of the number of customers at an arbitrary instant.

- Arrival theorem (open network):
  In an open network that is in steady state, a customer arriving at station j observes the system as if he observes it at an arbitrary instant.

- Arrival theorem (closed network):
  In a closed network with m customers that is in steady state, a customer arriving at station j observes the system as if he observes the network with (m-1) customers at an arbitrary instant.

Closed networks of M/M/1 queues: MVA

Mean queue length, mean response time?

$\lambda_m(i)$ arrival rate at station i,
$F_m(i)$ expected response time at station i,
$L_m(i)$ expected queue length at station i, if m customers in system

$$F_m(j) = \frac{1}{\mu_j} + L_{m-1}(j)\frac{1}{\mu_j}$$ Arrival theorem, FCFS

$$L_m(j) = \lambda_m(j) F_m(j)$$

Little's law

Another relation needed to determine $\lambda_m(j)$

Closed networks of M/M/1 queues: MVA

- Notation
  - *m* : number of customers in the system
  - π steady state distribution of Markov chain with transition matrix $R=(r_{ij})$:

$$\pi_i = \sum_{j=1}^{N} \pi_j \cdot r_{ji} \quad , \quad \sum_{i=1}^{N} \pi_i = 1$$

- $\lambda_m(i)$ is the arrival rate at station i, if m customers are circulating in the system
- Flow equations

$$\lambda_m(i) = \sum_{j=1}^{N} \lambda_m(j) \cdot r_{ji}$$

- Such that

$$\lambda_m(j) = \lambda_m \pi_j \qquad \lambda_m = \sum_{j=1}^{N} \lambda_m(j)$$

- λ_m # customers that are served in the system per time unit

Closed networks of M/M/1 queues: MVA

$\lambda_m(i)$ arrival rate at station i,
$F_m(i)$ expected response time at station i,
$L_m(i)$ expected queue length at station i, if m customers in system

$$\pi_i = \sum_{j=1}^{N} \pi_j \cdot r_{ji} \quad , \qquad \sum_{i=1}^{N} \pi_i = 1$$

$$\lambda_m(i) = \sum_{j=1}^{N} \lambda_m(j) \cdot r_{ji} \qquad \lambda_m(i) = \lambda_m \cdot \pi_i \quad , \quad \lambda_m = \sum_{i=1}^{N} \lambda_m(i)$$

- Little's law

$$m = \sum_{i=1}^{N} L_m(i) = \sum_{i=1}^{N} \lambda_m(i) \cdot F_m(i) = \lambda_m \cdot \sum_{i=1}^{N} \pi_i \cdot F_m(i)$$

- Consequence $\quad \lambda_m = m \cdot \left\{ \sum_{i=1}^{N} \pi_i \cdot F_m(i) \right\}^{-1}$

Closed networks of M/M/1 queues: MVA

$\lambda_m(i)$ arrival rate at station i,
$F_m(i)$ expected response time at station i,
$L_m(i)$ expected queue length at station i, if m customers in system

$$F_m(j) = \frac{1}{\mu_j} + L_{m-1}(j)\frac{1}{\mu_j},$$

$$L_m(j) = \lambda_m(j)F_m(j)$$

$$\lambda_m(i) = \lambda_m \cdot \pi_i = m \cdot \left\{\sum_{j=1}^{N} \pi_j \cdot F_m(j)\right\}^{-1} \cdot \pi_i$$

Closed networks of M/M/1 queues: MVA

calculates $\lambda_m(i)$, $F_m(i)$ and $L_m(i)$ for all m,i through recursion

Algorithm

- Determine solution π of the flow equations
- m:=1
- $F_1(i)=1/\mu_i$ for all i

R: recursive step

$$\lambda_m(i) = \lambda_m \cdot \pi_i = m \cdot \left\{ \sum_{j=1}^{N} \pi_j \cdot F_m(j) \right\}^{-1} \cdot \pi_i$$

$$L_m(i) = \lambda_m(i) \cdot F_m(i)$$

$$F_{m+1}(i) = \frac{1 + L_m(i)}{\mu_i}$$

m:=m+1 and go to R

Generalisations

- Approximate MVA

- QNA:
  W. Whitt (1983) The queueing network analyzer. Bell. Syst. Tech J 62, pp 2779 - 2815