



Stability and attraction domains of traffic equilibria in a day-to-day dynamical system formulation

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ABSTRACT

We formulate the traffic assignment problem from a dynamical system approach. All exogenous factors are considered to be constant over time and user equilibrium is being pursued through a day-to-day adjustment process. The traffic dynamics is represented by a recurrence function, which governs the system evolution over time. Equilibrium stability and attraction domain are then analyzed by studying the topological properties of the system evolution. Stability is important because unstable equilibrium is transient. Even for stable equilibrium, only points within its attraction domain are attracted to the equilibrium. We show that the attraction domain of a stable equilibrium is always open. Furthermore, its boundary is formed by trajectories towards unstable equilibria. Through an understanding of these properties, computation schemes can be devised to determine the ranges of the attraction domains, as demonstrated in this study. Once this is accomplished, a partition chart can be drawn on the state space where each part represents the attraction domain of an equilibrium point. We trust that charting the attraction domains of user equilibria, as presented in this paper, will open up innovative ways for transportation network management.

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1. Introduction

Ever since the notion of user equilibrium (UE) was proposed (Wardrop, 1952), it has become a cornerstone for traffic assignment analysis. At equilibrium, all travelers on the same origin–destination pair experience the same travel cost, while all unused routes have equal or higher travel costs. The equilibrium forms a stationary state because there is no alternative route with a lower cost and hence no incentive for travelers to switch routes. Past studies have primarily focused on the issues of equilibrium existence and uniqueness. The assumption is that if equilibrium exists, then it will also occur. This, of course, is an idealization. In fact, it was shown that quite the contrary could happen. Horowitz (1984) demonstrated that even for a well-behaved system whose equilibrium solution was known to exist, depending on the dynamic route adjustment process, the system might still fail to converge to equilibrium. Therefore, it is not sufficient to only ask whether equilibrium exists or not; it is equally important to ask whether and how the system can achieve equilibrium. In other words, the dynamic route adjustment processes of travelers in search of better routings constitute a key part of the analysis.

To address this problem further, we need to elaborate on the effects and properties of dynamic route adjustment processes. Studies on asymptotic stability, e.g. Smith (1979, 1984) and Watling (1999), deal with the “attractiveness” of an equilibrium state in this dynamic process. Stability is important in the sense that unstable equilibrium is unsustainable: both the equilibrium point and trajectories towards it are sensitive to disturbances; even a small perturbation will lead the system

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evolution away from the equilibrium point. For stable equilibrium, its stability can be classified as either global or local, i.e. whether its attraction domain covers the whole state space or not. For an equilibrium that possesses global stability, any initial state will dynamically evolve towards and eventually converge to the equilibrium. On the other hand, for an equilibrium that possesses only local stability, only initial states within its attraction domain will converge; initial states outside the attraction domain will not. Therefore, for traffic management, knowledge on the exact range of the attraction domain is essential, so as to direct the traffic evolution towards the desired equilibrium.

It is nonsensical to discuss the stability and attraction domain of equilibrium without referring to the underlying dynamical evolution. In particular, we consider daily commuters who make their trips from day to day. To model the evolution of traffic flow over time, it would be convenient if we know the stimulus of such temporal variation, i.e. why and how travelers change their travel choices from time to time. This stimulus defines the mechanism of traffic evolution over time. There are two approaches to this: a continuous-time formulation (Smith, 1984; Friesz et al., 1994; Zhang and Nagurney, 1996; Watling, 1999; Cho and Hwang, 2005; Mounce, 2006) and a discrete time formulation (Horowitz, 1984; Cantarella and Cascetta, 1995; Watling, 1999). In both approaches, the system evolves in a deterministic fashion so that all future states are known once the initial state is given. Alternatively, the traffic evolution can be formulated as a stochastic process (Cascetta, 1989; Davis and Nihan, 1993; Cantarella and Cascetta, 1995; Watling and Hazelton, 2003).

In this paper we focus on the deterministic evolution of traffic flow from day to day and we formulate it as a discrete time dynamical system. In Section 2, user equilibrium in the static traffic assignment and fixed point of the dynamical traffic system are defined and shown to be equivalent for both deterministic and stochastic user equilibrium. Section 3 addresses the stability and attraction domain of user equilibrium. It provides some topological analyses which are useful for the estimation of the equilibrium's attraction domain in Section 4. Although illustrated only with examples of low dimensions, the methods presented in this paper are applicable to network dynamics of higher dimensions. Finally, Section 5 gives some concluding remarks and topics for future research.

2. User equilibrium in a dynamical traffic system

In this section, we establish the relationship between the static user equilibrium in traffic assignment and the fixed point in day-to-day traffic dynamics. User equilibrium, by definition, signifies a steady state where travelers have no incentive of switching routes (when they make the trip again). If the dynamical traffic system is reasonably defined, the equilibrium should be a fixed (stationary) point in the dynamic process, and vice versa. That is, user equilibrium is time-invariant; if the dynamic process of traffic evolution reaches equilibrium it will remain there forever.

2.1. Traffic assignment and static user equilibrium

We consider a network with N origin–destination (OD) pairs. Each OD pair i ($i = 1, 2, \dots, N$) is connected by a set of routes, denoted as \mathbf{R}_i , with $m_i = |\mathbf{R}_i|$ and $M = \sum_{i=1}^N m_i$. Routes are numerated as $1, 2, \dots, m_1$ for routes in \mathbf{R}_1 , $m_1 + 1, m_1 + 2, \dots, m_1 + m_2$ for routes in $\mathbf{R}_2 \dots$, and $M - m_N + 1, M - m_N + 2, \dots, M$ for routes in \mathbf{R}_N . On day n , a demand of $d_i^{(n)}$ travelers on OD pair i make their route choices over the route set \mathbf{R}_i . A feasible route flow assignment is then represented by the M -vector $\mathbf{f}^{(n)} = [f_1^{(n)}, f_2^{(n)}, \dots, f_r^{(n)}, \dots, f_M^{(n)}]^T$, $\mathbf{f}^{(n)} \in \mathbf{D}^{(n)}$ where the feasible set $\mathbf{D}^{(n)}$ characterizes the demand constraints,

$$\mathbf{D}^{(n)} = \left\{ \mathbf{f}^{(n)} \in \mathbb{R}_+^M : \sum_{r \in \mathbf{R}_i} f_r^{(n)} = d_i^{(n)}, \quad \forall i = 1, 2, \dots, N \right\}. \quad (1)$$

In this paper we only consider the case where the travel demands are fixed and do not change over time. The superscript (n) in $d_i^{(n)}$ can then be removed; the feasible set $\mathbf{D}^{(n)} = \mathbf{D}$ is thus also fixed over time.

The M -vector of route costs (or disutilities), $\mathbf{c}^{(n)} = [c_1^{(n)}, c_2^{(n)}, \dots, c_r^{(n)}, \dots, c_M^{(n)}]^T$, is derived from the route cost function:

$$\mathbf{c}^{(n)} = \mathbf{c}(\mathbf{f}^{(n)}). \quad (2)$$

The route flow assignment $\mathbf{f}^{(n)}$ is termed *deterministic user equilibrium* (DUE) if

$$f_r^{(n)} > 0 \Rightarrow c_r(\mathbf{f}^{(n)}) \leq c_s(\mathbf{f}^{(n)}), \quad \forall r, s \in \mathbf{R}_i, r \neq s, \quad i = 1, 2, \dots, N. \quad (3)$$

The traffic state at DUE is considered “steady” because every traveler on the same OD pair experiences the same cost regardless of which route they choose; each unused route has an equal or higher cost. As such, the system offers no incentive for any traveler to switch route. Thus, traffic is at user equilibrium (Wardrop, 1952; Smith, 1979).

In the DUE formulation, travelers are assumed to have perfect information of route costs over the entire network, in contrast to the reality that travelers perceive travel costs at different precision levels. When perception error is considered, the same trip has different perceived travel costs for different travelers. If each traveler chooses the route with the least cost according to their perception, the principle of *stochastic user equilibrium* (SUE) is applied to model the resultant route choice pattern, as is customary in the literature (Sheffi, 1985).

SUE is often formulated with discrete choice models such as the logit model. For an individual traveler, perceived costs of the alternative routes are assumed to be identically and independently (i.i.d.) distributed Gumbel variates. The probability of

choosing route r is given as a function of the vector $\mathbf{C}^{(n)} = [C_1^{(n)}, C_2^{(n)}, \dots, C_r^{(n)}, \dots, C_M^{(n)}]^T$, where $C_r^{(n)}$ is the mean perceived cost of route r ,

$$p_r(\mathbf{C}^{(n)}) = \frac{1}{1 + \sum_{s \in \mathbf{R}_i, s \neq r} e^{\theta(C_r^{(n)} - C_s^{(n)})}}, \quad \forall r \in \mathbf{R}_i, \quad i = 1, 2, \dots, M, \tag{4}$$

where $\theta \geq 0$ is the dispersion parameter. Route flows are thus assigned as

$$\mathbf{f}^{(n)} = \mathbf{F}p(\mathbf{C}^{(n)}), \tag{5}$$

where \mathbf{F} is a constructed $M \times M$ diagonal matrix with $F_{rr} = d_i, \forall r \in \mathbf{R}_i, i = 1, 2, \dots, N$ and $p(\mathbf{C}^{(n)}) = \mathbf{p} = [p_1(\mathbf{C}^{(n)}), p_2(\mathbf{C}^{(n)}), \dots, p_r(\mathbf{C}^{(n)}), \dots, p_M(\mathbf{C}^{(n)})]^T$ is the M -vector of route choice probabilities. It is evident that $\mathbf{f}^{(n)} \in \mathbf{D}$; hence the feasibility requirement is fulfilled.

SUE is said to arise when for each route the mean perceived cost equals the actual cost, i.e.

$$\mathbf{C}^{(n)} = c(\mathbf{f}^{(n)}). \tag{6}$$

Combining (5) and (6) gives the SUE solution. At SUE, no traveler can improve their perceived travel cost by unilaterally changing route; therefore a traveler has no incentive to switch route and SUE represents a steady state of traffic assignment. We note in passing that such a SUE model, despite its name, is indeed a deterministic way of conducting traffic assignment; the network flow at SUE is deterministic rather than stochastic.

2.2. Dynamical traffic system and fixed point

The static state definition of user equilibrium as above is based on travelers' non-motivation to switch routes, in that user equilibrium brings no incentive for the travelers to change their current route choices (so as to minimize trip cost). However, to investigate the process of achieving equilibrium from a non-equilibrium state, we need to look at the dynamical system of traffic evolution over time, i.e. the day-to-day adjustment mechanism.

We denote $\mathbf{x}^{(n)}$ as the state identifier of traffic flows on day n . For instance, we can use the route flow vector as the state identifier, $\mathbf{x}^{(n)} = \mathbf{f}^{(n)}$; for SUE assignment, we can also use the route cost vector, $\mathbf{x}^{(n)} = \mathbf{C}^{(n)}$. The day-to-day traffic dynamics is defined by the following first-order recurrence equation:

$$\mathbf{x}^{(n)} = y(\mathbf{x}^{(n-1)}). \tag{7}$$

The function $y: \mathbf{D} \rightarrow \mathbf{D}$ maps the traffic state on day $n - 1$ to the state on day $n, n = 1, 2, \dots$; here \mathbf{D} is the feasible set of $\mathbf{x}^{(n)}$.

The recurrence function in (7) defines in a deterministic way the dynamical evolution over time from an initial state. Given the initial state $\mathbf{x}^{(0)} \in \mathbf{D}$, all future states $\mathbf{x}^{(n)}$ remain in \mathbf{D} and can be derived by iterating the recurrence function,

$$\mathbf{x}^{(n)} = \underbrace{y(y(\dots y(\mathbf{x}^{(0)} \dots)))}_n = y^{(n)}(\mathbf{x}^{(0)}). \tag{8}$$

The evolution path of an initial state $\mathbf{x}^{(0)}$ forms a trajectory:

$$\mathbf{x}^{(0)}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \dots, \mathbf{x}^{(n)} \dots \tag{9}$$

Strictly speaking, future states should be written as $\mathbf{x}^{(n)} | \mathbf{x}^{(0)}, n = 1, 2, \dots$ since they intrinsically depend on the initial state $\mathbf{x}^{(0)}$. For notational simplicity, however, we omit the specification of the initial state unless confusion is likely to arise.

A point \mathbf{x}^* is called a *fixed point* if

$$\mathbf{x}^* = y(\mathbf{x}^*). \tag{10}$$

That is, the dynamical evolution as defined by (7) maps the fixed point to itself. Generalizing this property over time, we have

$$\mathbf{x}^* = y^{(n)}(\mathbf{x}^*), \quad n = 1, 2, \dots \tag{11}$$

Therefore, the fixed point is stationary; a trajectory started at the fixed point will remain there for all future days.

The property of being time-invariant is shared by user equilibrium in traffic assignment and fixed point of the dynamical traffic system. Indeed the two concepts describe the same phenomenon (i.e. time-invariant traffic flow), but from different perspectives: the former from the motivational observation of travelers' route switching behavior, the latter from the mechanism of traffic flow transformation. If both concepts are reasonably defined and in accordance with travelers' behavior, they should coincide with and be equivalent to each other, as we shall see in the models to follow. From now on, we call fixed point and user equilibrium interchangeably and notate them as \mathbf{x}^* .

2.3. Dynamical traffic system with DUE

We consider the following recurrence function (Cantarella and Cascetta, 1995):

$$\mathbf{f}^{(n)} = \mathbf{Q}^{(n)} \mathbf{W}^{(n)} \mathbf{f}^{(n-1)} + (\mathbf{I} - \mathbf{W}^{(n)}) \mathbf{f}^{(n-1)}, \tag{12}$$

where $\mathbf{W}^{(n)} = W(\mathbf{f}^{(n-1)})$ is a diagonal $M \times M$ matrix whose entry $w_{kk}^{(n)} \in [0, 1]$ represents the probability that travelers on route k for day $n - 1$ will reconsider their route choices for day n . The block diagonal $M \times M$ matrix $\mathbf{Q}^{(n)} = Q(\mathbf{f}^{(n-1)})$ contains one $m_i \times m_i$ block $\mathbf{Q}_{ij}^{(n)}$ for each OD pair i ; an entry $q_{kj}^{(n)} \in [0, 1]$ for $k, j \in \mathbf{R}_i$ represents the conditional probability that travelers reconsidering their route choices on j for day $n - 1$ will switch to route k for day n . Note that a positive $q_{jj}^{(n)}$ implies that some users on route j for day $n - 1$, even if reconsidering, will continue to use the same route for day n . The feasibility constraint (flow conservation) is satisfied by specifying

$$\sum_{k \in \mathbf{R}_i} q_{kj}^{(n)} = 1, \quad \forall j \in \mathbf{R}_i, \forall i. \tag{13}$$

Then for the feasible set in (1), we can establish

$$\mathbf{f}^{(n-1)} \in \mathbf{D} \Rightarrow \mathbf{f}^{(n)} \in \mathbf{D}. \tag{14}$$

Therefore the recurrence function in (12) is a mapping from \mathbf{D} to \mathbf{D} .

Fixed point of the dynamical system (12) is represented by:

$$\mathbf{f}^* = \mathbf{Q}(\mathbf{f}^*)W(\mathbf{f}^*)\mathbf{f}^* + (\mathbf{I} - W(\mathbf{f}^*))\mathbf{f}^*. \tag{15}$$

This fixed point is equivalent to user equilibrium as defined in (3) if the following three requirements are enforced:

$$w_{kk}^{(n)} > 0, \quad \forall k; \tag{16}$$

$$c_k^{(n-1)} \geq c_j^{(n-1)} \Rightarrow q_{kj}^{(n)} = 0, \quad \forall k, j \in \mathbf{R}_i, k \neq j, \forall i; \tag{17}$$

$$c_k^{(n-1)} < c_j^{(n-1)} \Rightarrow q_{kj}^{(n)} > 0, \quad \forall k, j \in \mathbf{R}_i, \forall i. \tag{18}$$

Condition (16) requires that a positive number of users will reconsider their route choices over time. If nobody does so (i.e. $w_{kk}^{(n)} = 0$ for all k) then any point is a fixed point but not necessarily a user equilibrium. Condition (17) says that travelers will not switch to a route with equal or higher cost; condition (18) says that if there is a cost difference, part of the flow on the higher cost route (if non-zero) will switch to the lower cost route. From a behavioral point of view, all the three conditions above can be reasonably assumed. A proof of the equivalence can be found in Appendix A.

Example 1. (Lo and Bie, 2006) Consider a network with one OD pair connected by two routes, and a fixed demand of one unit. The flows on the two routes are denoted by $\mathbf{f} = [f_1, f_2]^T$, with the feasible set defined as $D = \{\mathbf{f} : 0 \leq f_1 \leq 1, f_2 = 1 - f_1\}$. The travel cost functions are linear and separable: $c_1(f_1) = 0.6f_1 + 0.4$ for route 1 and $c_2(f_2) = 0.4f_2 + b$ for route 2, where $b \in (0, 1)$ is a parameter to be varied for illustration purposes. Because demand is fixed, there is only one degree of freedom for the route flows. If we denote $x = f_1 \in [0, 1]$ as the flow on route 1, then the flow on route 2 is determined as $f_2 = 1 - x$. Therefore a single variable x is sufficient to specify the flow dynamics and qualifies as state identifier. Let $g(x) = c_1(x) - c_2(1 - x) = x - b$ be the travel cost difference between the two routes, then solving the equation $g(x) = 0$ gives the equilibrium flow $x^* = b$.

We specify the day-to-day dynamics by setting $\mathbf{W}^{(n)} = \mathbf{I}$ and $\mathbf{Q}^{(n)} = \begin{bmatrix} 1 - p_1^{(n)} & p_2^{(n)} \\ p_1^{(n)} & 1 - p_2^{(n)} \end{bmatrix}$ in (12). We further define $p_1^{(n)} = \min\{1, \alpha[g(x^{(n-1)})]_+\}$ and $p_2^{(n)} = \min\{1, \alpha[-g(x^{(n-1)})]_+\}$, where $[z]_+ = \max\{0, z\}$. In this setting, all travelers reconsider their previous route choices. For travelers currently on route 1, a proportion of p_1 will switch to route 2; for travelers currently on route 2, a proportion of p_2 will switch to route 1. If currently $c_1(x) > c_2(1 - x)$ then $g(x) > 0$, $p_1 > 0$ and $p_2 = 0$; if $c_1(x) < c_2(1 - x)$ then $g(x) < 0$, $p_1 = 0$ and $p_2 > 0$; and if $c_1(x) = c_2(1 - x)$ then $g(x) = 0$, $p_1 = p_2 = 0$ and it is equilibrium. The dynamical system here satisfies the conditions set by (16)–(18). Therefore fixed point of the system corresponds to user equilibrium. Taking the settings here into (12), we have the recurrence function of x as

$$y(x) = \begin{cases} \min\{x + \alpha(1 - x)(b - x), 1\}, & x \leq b; \\ \max\{x - \alpha x(x - b), 0\}, & x > b. \end{cases}$$

For the specific case of $b = 0.4$ and $\alpha = 2.5$, the recurrence function $x^{(n+1)} = y(x^{(n)})$ is shown as the curve in Fig. 1. The diagonal line $y = x$ is drawn to help illustrate the system evolution over time. Consider the evolution from an arbitrary point x , we start by drawing a vertical line through (x, x) and it intersects with the curve at $(x, y(x))$. We then draw a horizontal line from $(x, y(x))$, which intersects with the diagonal line at $(y(x), y(x))$. This gives, on both the horizontal and vertical axes, the value of the system after one step. Subsequently, we draw a vertical line from $(y(x), y(x))$ to reach the point $(y(x), y^{(2)}(x))$ and then a horizontal line to reach $(y^{(2)}(x), y^{(2)}(x))$, which gives the system value after two steps of evolution. By repeatedly constructing these vertical and horizontal lines, we can generate all future states of the evolution from x .

2.4. Dynamical traffic system with SUE

We consider the following dynamical traffic system (Watling, 1999):

$$\begin{cases} \mathbf{C}^{(n)} = \beta c(\mathbf{f}^{(n-1)}) + (1 - \beta)\mathbf{C}^{(n-1)}, \\ \mathbf{f}^{(n)} = \mathbf{F}p(\mathbf{C}^{(n)}). \end{cases} \tag{19}$$

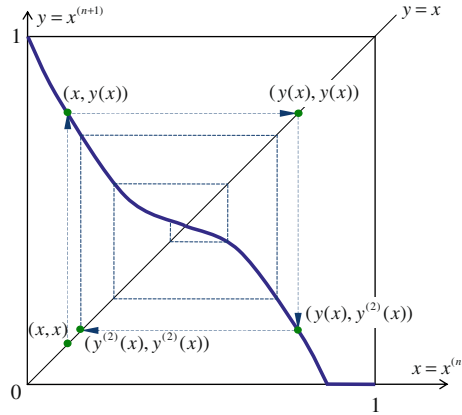


Fig. 1. Recurrence function in Example 1.

The transformation of traffic flow from one day to the next day can be illustrated as in Fig. 2 (where demand can be either fixed or dependent on the perceived cost). On day n , the mean perceived cost $\mathbf{C}^{(n)}$ is updated as the weighted average of the previous day’s perceived cost $\mathbf{C}^{(n-1)}$ and actual cost. The logit assignment model is then applied, leading to day n ’s traffic flow $\mathbf{f}^{(n)}$ and actual travel cost $\mathbf{c}^{(n)}$. Updating $\mathbf{C}^{(n)}$ with $\mathbf{c}^{(n)}$ gives $\mathbf{C}^{(n+1)}$, the start of another day in the dynamic process.

The parameter β ($0 < \beta \leq 1$) here represents the weight of the actual cost in updating the perceived cost. A small β describes the behavior with a strong habitual tendency, while a big β means that users rely more on recent information (in our case the previous day’s actual travel costs). In the extreme case of $\beta = 1$, all past memories are abandoned and the cost experienced on the previous day is taken directly as the perceived cost of the current day.

Using the vector of mean perceived costs as the state identifier, the system can be written as

$$\mathbf{C}^{(n)} = \beta \mathbf{c}(\mathbf{Fp}(\mathbf{C}^{(n-1)})) + (1 - \beta)\mathbf{C}^{(n-1)} \tag{20}$$

Fixed point of the dynamical system can be derived as

$$\mathbf{C}^* = \mathbf{c}(\mathbf{Fp}(\mathbf{C}^*)) \tag{21}$$

This is exactly the SUE condition. Therefore an equivalence has been established between the SUE solution as defined in (5) and (6), and the fixed point of the dynamical system (19).

The dynamical traffic system with SUE can be also formulated in ways other than (19), e.g. Cantarella and Cascetta (1995). The dynamical system model in (19) is rather “radical” in that all users reconsider their route choices on a daily basis. Modification can be made to (19) such that only a proportion of travelers reevaluate their route choice each day.

Example 2. (Watling, 1999) Consider a network with one OD pair connected by three routes, and a fixed demand of two units. The cost functions for the three routes are as follows:

$$c_1(\mathbf{f}) = f_1 + 3f_2 + 1, \quad c_2(\mathbf{f}) = 2f_1 + f_2 + 2, \quad c_3(\mathbf{f}) = f_3 + 6. \tag{22}$$

Values for parameters in the dynamical system are $\theta = 1$ and $\beta = 0.2$. Due to the asymmetric nature of the cost functions, multiple equilibria exist,

$$\mathbf{f}_I^* = [1.75, 0.15, 0.10]^T, \mathbf{f}_{II}^* = [0.77, 1.03, 0.20]^T, \mathbf{f}_{III}^* = [0.22, 1.59, 0.19]^T. \tag{23}$$

Since the demand is fixed, the system only has two degrees of freedom. To represent the dynamical system we only need two variables. We can use the route cost differences as the state identifier: $(g_1, g_2) = (c_1 - c_2, c_1 - c_3)$. Then, by substituting the flows of the equilibria in (23) into the routes’ cost functions in (22), we obtain:

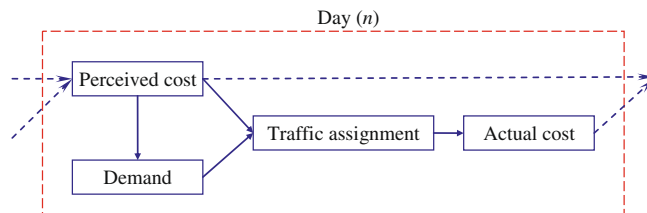


Fig. 2. Dynamical traffic system with SUE.

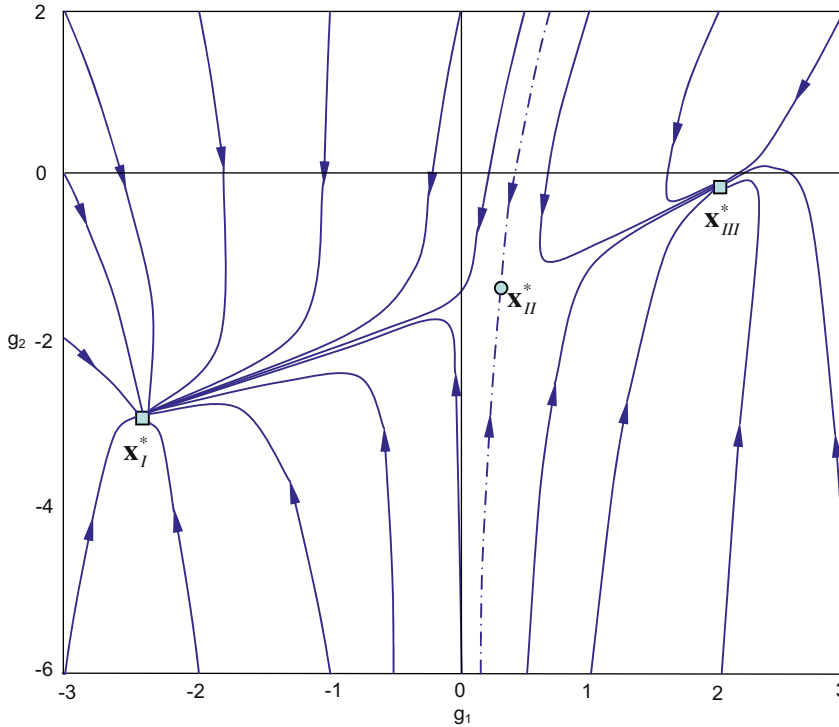


Fig. 3. Phase portrait for Example 2.

$$(g_1, g_2)_I^* = (-2.45, -2.89), \quad (g_1, g_2)_{II}^* = (0.30, -1.34), \quad (g_1, g_2)_{III}^* = (1.95, -0.19).$$

A phase portrait of this dynamical system using the (g_1, g_2) dimensions is shown in Fig. 3. The phase portrait is a plot of typical trajectories in the state space. It can only be visualized for low dimension systems. We note that the phase portrait here is only drawn for illustration purpose. In practice, where the dynamical traffic systems have much higher dimensions, we do not possess the knowledge of such phase portraits.

3. Stability and attraction domain of user equilibrium

Stability and attraction domain of user equilibrium are studied in this section. Stability defines the “sustainability” of the equilibrium; a temporary, small perturbation does not lead to a large deviation from the stable equilibrium and the equilibrium can recover itself within time. The attraction domain defines the collection of all states that will evolve towards the user equilibrium over time. Therefore the attraction domain characterizes the attractiveness of the user equilibrium from the global state space (i.e. the feasible set).

3.1. Topological definitions

Some topological definitions are necessary for the subsequent explorations on stability and attraction domain. For the discussion to be of any interest in reality, the state space \mathbf{D} is considered as bounded; any point (or state) is an element of \mathbf{D} and any set is a subset of \mathbf{D} . The dimension of the space is arbitrarily set as l .

- (a) The *distance* between two states \mathbf{x} and \mathbf{y} , denoted as $\|\mathbf{x} - \mathbf{y}\|$, is the Euclidean metric, i.e. $\|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^l (x_i - y_i)^2}$. Obviously we have $\|\mathbf{x} - \mathbf{x}\| = 0$, $\|\mathbf{x} - \mathbf{y}\| \geq 0$, $\|\mathbf{x} - \mathbf{y}\| = \|\mathbf{y} - \mathbf{x}\|$, and $\|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{z}\| \geq \|\mathbf{x} - \mathbf{z}\|$ for any \mathbf{z} .
- (b) A *neighborhood* of \mathbf{x} is the collection of all states whose distance from \mathbf{x} is less than a specified positive value, i.e. the set $\mathbf{H}(\mathbf{x}, \delta) = \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| < \delta\}$ is the neighborhood of \mathbf{x} at distance level δ ($\delta > 0$). Hence the neighborhood can be considered as an l -ball of radius δ .
- (c) The *complement* of a set \mathbf{S} , denoted as $\bar{\mathbf{S}}$, is the collection of all states that are not in \mathbf{S} , i.e. $\bar{\mathbf{S}} = \{\mathbf{x} : \mathbf{x} \notin \mathbf{S}\}$.
- (d) A set \mathbf{S} is *open* if any point \mathbf{x} in \mathbf{S} has a neighborhood that lies entirely in \mathbf{S} , i.e. $\forall \mathbf{x} \in \mathbf{S}, \exists \delta > 0 : \mathbf{H}(\mathbf{x}, \delta) \subset \mathbf{S}$; a set \mathbf{S} is *closed* if any point \mathbf{x} outside \mathbf{S} has a neighborhood that lies entirely outside \mathbf{S} , i.e. $\forall \mathbf{x} \in \bar{\mathbf{S}}, \exists \delta > 0 : \mathbf{H}(\mathbf{x}, \delta) \subset \bar{\mathbf{S}}$. Therefore, if \mathbf{S} is open then $\bar{\mathbf{S}}$ is closed; and vice versa. The state space, or any empty set, is considered as both open and closed. Of course there are also sets that are neither open nor closed.

- (e) The *closure* of a set S , denoted as $\overset{\circ}{S}$, is the smallest closed set containing S . If S is itself closed, then $\overset{\circ}{S} = S$. The *boundary* of a set S , denoted as ∂S , is the intersection of its closure and the closure of its complement, i.e. $\partial S = \overset{\circ}{S} \cap \overset{\circ}{\bar{S}}$. The boundary of \bar{S} is therefore identical to the boundary of S , i.e. $\partial \bar{S} = \partial S$.
- (f) A set S is said to be *invariant* with respect to the dynamical system (7) if a dynamical evolution started within S shall forever remain in S , i.e. $\mathbf{x}^{(0)} \in S \Rightarrow \mathbf{x}^{(n)} \in S, \forall n = 1, 2, \dots$. Furthermore, S is said to be *strictly invariant* if S is invariant and a dynamical evolution started outside S shall forever stay outside of S , i.e. $\mathbf{x}^{(0)} \notin S \Rightarrow \mathbf{x}^{(n)} \notin S, \forall n = 1, 2, \dots$.

Based on these definitions, a set is open if its boundary is not part of the set; a set is closed if its boundary is wholly in the set. The closure of an open set is then the union of itself and its boundary. Consider a point \mathbf{x} on the boundary of an open set S , $\mathbf{x} \in \partial S$; it satisfies that:

- (i) $\mathbf{x} \notin S$; and,
- (ii) $H(\mathbf{x}, \delta) \cap S \neq \emptyset, \forall \delta > 0$.

Here property (ii) means that any neighborhood of \mathbf{x} has a non-empty intersection with S . This property is very useful for characterizing points on the boundary of an open set.

Invariance of a set means that any dynamical evolution started in the set will not “escape”. Therefore all future states are contained in the set and the set can “maintain” itself over time. Strict invariance further requires that any dynamical evolution started outside the set remains forever outside the set; that is, any evolution that is now in the set must have started in the set. Thus the set is invariant even under the inverse dynamical system (if the system is invertible). For a trajectory started from any point in the state space, it lies either entirely inside the strictly invariant set or entirely outside the set. Hence there is no trajectory crossing the boundary of a strictly invariant set.

3.2. Definitions of asymptotic stability

Various definitions of stability have been made in the literature. In this study, we adopt the following definitions:

Convergence: The fixed point \mathbf{x}^* is *convergent* if there is a neighborhood of \mathbf{x}^* where all points converges to \mathbf{x}^* , i.e. $\exists \delta > 0 : \mathbf{x}^{(0)} \in H(\mathbf{x}^*, \delta) \Rightarrow \lim_{n \rightarrow +\infty} \mathbf{x}^{(n)} = \mathbf{x}^*$.

Stability: The fixed point \mathbf{x}^* is *stable* if a trajectory can be made to remain arbitrarily close to \mathbf{x}^* by starting sufficiently close to \mathbf{x}^* , i.e. $\forall \varepsilon > 0, \exists \delta > 0 : \mathbf{x}^{(0)} \in H(\mathbf{x}^*, \delta) \Rightarrow \|\mathbf{x}^{(n)} - \mathbf{x}^*\| < \varepsilon, n = 1, 2, \dots$.

Asymptotic stability: The fixed point \mathbf{x}^* is *asymptotically stable* if it is both stable and convergent.

Convergence is concerned only with asymptotic (infinite time) properties. It means that as a result of the dynamical evolution, all points within the neighborhood of fixed point will eventually come to the fixed point. Stability, on the other hand, is concerned with the adjacency of a nearby trajectory. By starting from a given neighborhood, all future states of the system must be arbitrarily close to the fixed point.

We show the difference between convergence and stability by two simple dynamical systems. The first is a one-dimensional system: $x^{(n+1)} = x^{(n)}, x \in \mathbb{R}$. Here any point x is a stable but non-convergent fixed point. It is stable because we can choose $\delta \triangleq \varepsilon$ and then the conditions in the stability definition hold. It is non-convergent because any dynamical evolution around the fixed point will not converge to the fixed point, no matter how close to the fixed point the evolution has started.

Next, we consider the following one-dimensional system:

$$x^{(n+1)} = \begin{cases} \frac{1}{x^{(n)}} & \text{if } x^{(n)} \neq 0 \text{ and } \frac{1}{x^{(n)}} \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

The fixed point $x^* = 0$ is convergent but not stable. It is convergent because an evolution started from any point (except ± 1) converges to the fixed point, at most in two steps of evolution and some in just one step. It is unstable because for a given $\varepsilon = 0.1$, no matter how δ is chosen (e.g. $\delta = 0.00 \dots 01$), we can always find an integer k big enough such that $1/k < \delta$ and $x^{(0)} = 1/k \in H(x^*, \delta) \Rightarrow x^{(1)} = k > \varepsilon$. Therefore there is no neighborhood of x^* that can maintain a distance closer than the given ε . We also notice here that the fixed points $x^* = \pm 1$ are neither stable nor convergent.

The recurrence function in the above system is rather peculiar. It is not continuous in the region $[-1, 1]$, where some points are mapped to zero while others are mapped to their reciprocals. Denote the recurrence function here as $x^{(n+1)} = f(x^{(n)})$; the dynamical system $y^{(n+1)} = g(y^{(n)}) = f(f(y^{(n)}))$ exhibits much simpler behavior:

$$y^{(n+1)} = \begin{cases} 1 & \text{if } y^{(n)} = 1, \\ -1 & \text{if } y^{(n)} = -1, \\ 0 & \text{otherwise.} \end{cases}$$

Here the fixed points $x^* = \pm 1$ are still unstable and non-convergent, but the fixed point $x^* = 0$ is asymptotically stable. The different stability results of the fixed point $x^* = 0$ in the two systems are caused by the difference of the recurrence function around the fixed point, one discontinuous and the other continuous, as explained by the proposition below.

Proposition 1. Consider the dynamical system $\mathbf{x}^{(n+1)} = y(\mathbf{x}^{(n)})$ and the fixed point $\mathbf{x}^* = y(\mathbf{x}^*)$. If $y(\cdot)$ is continuous and \mathbf{x}^* is convergent, then \mathbf{x}^* is stable.

Proof. See Appendix B. \square

This proposition shows that for a continuous dynamical system, a convergent fixed point must also be stable (and therefore asymptotically stable). Continuity of the recurrence function is then a desirable property for the dynamical system. However, we note that even for a continuous dynamical system, a stable fixed point may well be non-convergent (as in the first system above).

3.3. Sufficient conditions for asymptotic stability

Asymptotic stability encompasses both the properties of convergence and stability. If the traffic system is currently at an asymptotically stable fixed point and a small deviation is inflicted, the system evolution will not diverge but shall return to the same fixed point over time. Therefore, an asymptotically stable fixed point is not affected by small perturbations and is capable of recovery. As such asymptotic stability is a local property; it pertains only to the dynamical behavior in a neighborhood of the fixed point. Therefore, stability (or instability) can be verified by investigating the local behavior of the dynamical system (7) around the fixed point \mathbf{x}^* (such as the Jacobian matrix). We now state Lyapunov’s stability theorem (Smith, 1984).

Lyapunov Theorem. Consider the fixed point $\mathbf{x}^* = y(\mathbf{x}^*)$ of the dynamical system (7), if there exists a continuously differentiable function $V(\mathbf{x}) : \mathbf{S} \rightarrow \mathbb{R}$ in a neighborhood \mathbf{S} of \mathbf{x}^* such that

- (i) $V(\mathbf{x}) > V(\mathbf{x}^*)$, $\forall \mathbf{x} \in \mathbf{S}$ and $\mathbf{x} \neq \mathbf{x}^*$, and
- (ii) $V(y(\mathbf{x})) \leq V(\mathbf{x})$, $\forall \mathbf{x} \in \mathbf{S}$,
then \mathbf{x}^* is stable. Moreover, if
- (iii) $V(y(\mathbf{x})) < V(\mathbf{x})$, $\forall \mathbf{x} \in \mathbf{S}$ and $\mathbf{x} \neq \mathbf{x}^*$,

then \mathbf{x}^* is asymptotically stable.

Example 1a. Consider the network in Example 1 with $b = 0.4$ and $\alpha = 2.5$. We can create the Lyapunov function as $V(x) = (x - 0.4)^2$. It is then not difficult to verify that the condition (i) above holds for all $x \in [0, 1]$ while conditions (ii) and (iii) holds for any $x \in (0.2, 0.8)$. Therefore the equilibrium $x^* = 0.4$ is asymptotically stable.

Example 2a. Consider the network in Example 2. Watling (1999) demonstrated the existence of such a Lyapunov function in a neighborhood for both \mathbf{g}_i^* and \mathbf{g}_{ij}^* . Therefore these two equilibria are stable. The existence of the Lyapunov function presents only a sufficient condition for stability; not having an explicit Lyapunov function does not necessarily mean that the equilibrium is unstable. So this method does not tell whether \mathbf{g}_{ij}^* is stable or not. However, analysis on the Jacobian matrix at \mathbf{g}_{ij}^* shows that it is indeed unstable.

3.4. The attraction domain of an asymptotically stable equilibrium

The attraction domain (or attraction basin) for a fixed point \mathbf{x}^* , denoted as $\mathbf{B}(\mathbf{x}^*)$, is the set of all states that will dynamically evolve to \mathbf{x}^* :

$$\mathbf{B}(\mathbf{x}^*) = \{\mathbf{x}^{(0)} \in \mathbf{D} : \lim_{n \rightarrow +\infty} \mathbf{x}^{(n)} = \mathbf{x}^*\}. \tag{24}$$

The attraction domain is non-empty even for unstable fixed point, for, at least, $\mathbf{x}^* \in \mathbf{B}(\mathbf{x}^*)$. If \mathbf{x}^* is asymptotically stable then there is a neighborhood of \mathbf{x}^* wherein every point is attracted to \mathbf{x}^* . This means that there exists a $\delta > 0$ such that $\mathbf{H}(\mathbf{x}^*, \delta) \subset \mathbf{B}(\mathbf{x}^*)$. To this end the asymptotic stability of a fixed point can also be defined by its attraction domain: the fixed point is asymptotically stable if and only if its attraction domain contains at least one of its neighborhoods. The attraction domain of an unstable equilibrium does not contain any of its neighborhoods.

The ideal situation of global asymptotic stability is defined as the case where the attraction domain covers the entire state space, i.e. $\mathbf{B}(\mathbf{x}^*) = \mathbf{D}$; or, equivalently,

$$\lim_{n \rightarrow +\infty} \mathbf{x}^{(n)} = \mathbf{x}^*, \quad \forall \mathbf{x}^{(0)} \in \mathbf{D}. \tag{25}$$

That is, the dynamical evolution started from any point in the state space will converge to the equilibrium. However, global asymptotic stability is not an intrinsic property of equilibrium. A simple counterexample of this is the case of multiple equilibria. If we denote $\{\mathbf{x}^*\}$ as the set of equilibrium points then

$$\mathbf{B}(\mathbf{x}_i^*) \cap \mathbf{B}(\mathbf{x}_j^*) = \Phi, \quad \forall \mathbf{x}_i^*, \mathbf{x}_j^* \in \{\mathbf{x}^*\}, \mathbf{x}_i^* \neq \mathbf{x}_j^*. \tag{26}$$

The mutual exclusiveness in (26) directly follows the definition of attraction domain. A point cannot converge to more than one equilibrium point and therefore must be in one attraction domain or another. The state space is thus divided into mutually exclusive regions, each of which represents the attraction domain of a fixed point.

When stability of a fixed point is not global, only points inside its attraction domain are attracted to the fixed point. Any dynamical evolution which started outside its attraction domain will not evolve towards the fixed point (but to another fixed point). It is therefore important and useful to identify the exact range of a fixed point's attraction domain, especially for the purpose of traffic management over time. To do so we first present some analyses on the properties of the attraction domain.

We consider the fixed point $\mathbf{x}^* = y(\mathbf{x}^*)$ of the dynamical system (7) and its attraction domain $\mathbf{B}(\mathbf{x}^*)$. First we notice that $\mathbf{B}(\mathbf{x}^*)$ is strictly invariant. That is, if $y^{(n)}(\mathbf{x}^{(0)}) \in \mathbf{B}(\mathbf{x}^*)$ for some $n = 0, 1, 2, \dots$, then $y^{(n)}(\mathbf{x}^{(0)}) \in \mathbf{B}(\mathbf{x}^*)$ for all n ; a trajectory is either entirely inside $\mathbf{B}(\mathbf{x}^*)$ or entirely outside $\mathbf{B}(\mathbf{x}^*)$. Now if we assume that the following conditions are true:

- (I) the function y in (7) is continuous,
- (II) every trajectory converges to a fixed point (one or another),
- (III) \mathbf{x}^* is asymptotically stable,

then the following properties hold for $\mathbf{B}(\mathbf{x}^*)$ and its boundary, $\partial\mathbf{B}(\mathbf{x}^*)$:

Proposition 2. $\mathbf{B}(\mathbf{x}^*)$ is open.

Proof. See Appendix C. \square

Proposition 3. $\partial\mathbf{B}(\mathbf{x}^*)$, if non-empty, is invariant.

Proof. See Appendix D. \square

Proposition 4. $\partial\mathbf{B}(\mathbf{x}^*)$, if non-empty, is formed by trajectories towards unstable fixed points.

Proof. See Appendix E. \square

The three propositions above are instrumental for devising computational schemes to determine the attraction domains of traffic equilibria.

Example 1b. Consider the network in Example 1. The dynamical system has a unique fixed point at $x^* = 0.4$. However, its attraction domain is not the global state space. This is not because there exists another fixed point but because not every trajectory converges to a fixed point (Assumption III). As shown in Fig. 4, there are two periodic cycles, both of period 2: $\{0.121, 0.734\}$ and $\{0, 1\}$. As illustrated by the phase portrait in Fig. 5, the unstable 2-cycle $\{0.121, 0.734\}$ actually forms the boundary of $\mathbf{B}(x^* = 0.4)$, i.e. the attraction domain for $x^* = 0.4$ is the open region $(0.121, 0.734)$. All trajectories within this

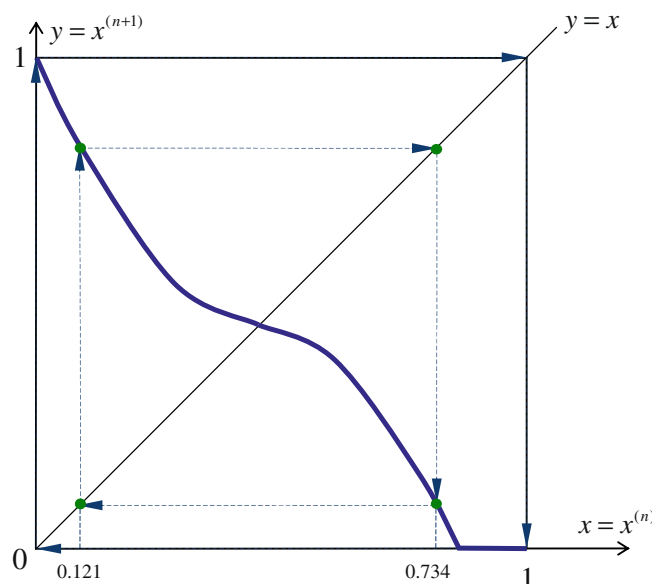


Fig. 4. Periodic cycles in Example 1: $\{0.121, 0.734\}$, $\{0, 1\}$.

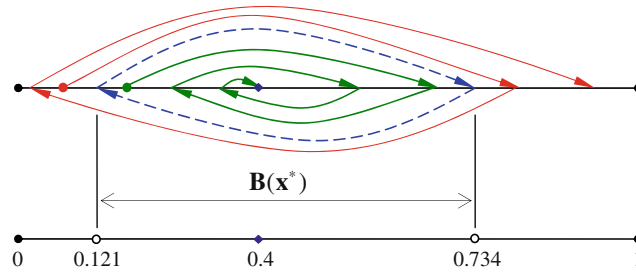


Fig. 5. Phase portrait for Example 1.

region are attracted to the fixed point. The trajectory from $x^{(0)} = 0.121$ will repeat periodically at intervals of two. All trajectories outside the region $[0.121, 0.734]$ converge to the 2-cycle $\{0, 1\}$.

In this example, there are non-equilibrium attractors in the dynamical system. Generally, an *attractor* of the dynamical system is defined as an invariant set where all elements communicate with each other. It can be classified into three categories: fixed point (one element), periodic cycle (a countable number of elements) and chaos (an uncountable number of elements). Similarly, stability and attraction domain can be defined for all types of attractors (Bie, 2008). In the above example, we show that $\partial B(x^*)$ is formed by trajectories of unstable attractors. The study of cycles and chaos exceeds the scope of this paper. From now on we only consider the case where Assumptions (I–III) are all true.

Because each trajectory ends at a fixed point and each fixed point has its own attraction domain, the state space can be partitioned into these mutually exclusive and collectively exhaustive attraction domains,

$$\cup_{x_i^* \in \{x^*\}} B(x_i^*) = D. \tag{27}$$

We can then draw a partition chart of the state space, where each part represents the attraction domain a fixed point. Once this is done, given the location of an initial state, we can immediately tell which fixed point the trajectory from this initial state will end up at. This is illustrated by the following example.

Example 2b. Consider the network in Example 2. As implied by the phase portrait in Fig. 3, each of the equilibria g_i^* and g_{III}^* is attractive from at least a local neighborhood; thus they are both stable. In contrast, g_{II}^* only attracts points on the dotted curve in Fig. 3. There is no neighborhood of g_{II}^* that g_{II}^* is attractive from; g_{II}^* is therefore unstable. As stated in Proposition 2, the attraction domains for the two stable equilibria, g_i^* and g_{III}^* , are both open. Their common boundary is formed by the trajectories towards the unstable equilibrium g_{II}^* (Proposition 4),

$$\partial B(g_i^*) = \partial B(g_{III}^*) = B(g_{II}^*).$$

As an attraction domain $B(g_i^*)$ is itself invariant, and so are $\partial B(g_i^*)$ and $\partial B(g_{III}^*)$ (Proposition 3). Furthermore, we can see that the state space can be partitioned into three regions (Fig. 6), each representing an attraction domain. Note here that the state space is not limited to the rectangle $\{-3 \leq g_1 \leq 3, -6 \leq g_2 \leq 2\}$.

4. Estimation of the attraction domain

The Lyapunov function $V(x) : S \rightarrow \mathbb{R}$ has been used to estimate the attraction domain for an asymptotically stable fixed point. We should first point out that the domain of the Lyapunov function, S , is not always an estimate for $B(x^*)$; a further condition is necessary to ensure convergence, namely, that S is invariant. If this condition is not satisfied, then there is no guarantee that the trajectory starting in S shall remain forever in S . Once the trajectory leaves S , the property of $V(y(x)) < V(x)$ may no longer hold and then the argument of $V(x)$ decreasing to $V(x^*)$ is not valid. This is the case for the asymptotically stable equilibrium in Example 1; domain of the Lyapunov function is $S = \{x: 0.2 < x < 0.8\}$ but it does not give an estimate of the attraction domain $B(x^* = 0.4) = \{x: 0.121 < x < 0.734\}$. The simplest way to ensure the invariant property of S is to estimate $B(x^*)$ by an invariant subset of S , e.g. $\Omega = \{x \in S: V(x) \leq b\}$ where b takes a positive value. When S is invariant, every trajectory in it converges to x^* . Therefore, an invariant S forms a subset of $B(x^*)$ and provides an estimate of $B(x^*)$.

The grid-search method has been proposed where a Lyapunov function can be systematically generated (Watling, 1999). A grid-search is then performed to maximally expand the region of S wherever all conditions in the Lyapunov Theorem remain valid. One limitation of estimating $B(x^*)$ through a Lyapunov function is that the output is only a subset S of $B(x^*)$. It remains unknown how extensively S covers $B(x^*)$. The region of $B(x^*)$ that is not covered by S can be much larger than S itself. Moreover, it remains uncertain whether a point outside S belongs to $B(x^*)$ or not.

It is therefore desirable to cover a broader region of $B(x^*)$ in the estimation result. In the following, we devise two methods to this end. The first method, based on the result of Propositions 2 to 4, is constructed by tracing back from the equilibria, which, if successfully performed, gives the full range of the attraction domains. The second method is constructed by sam-

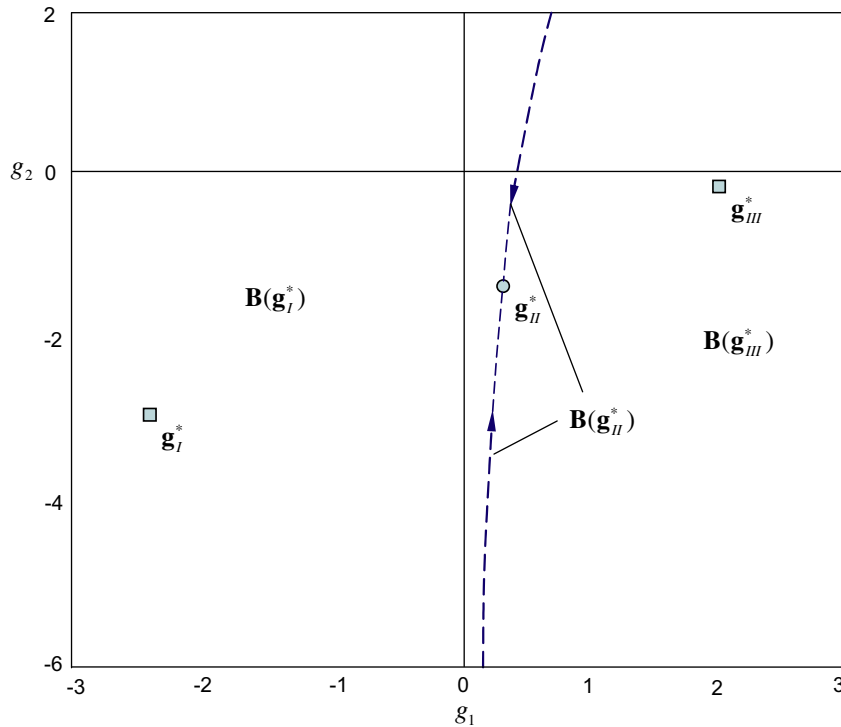


Fig. 6. Partition chart of the state space in Example 2.

pling and simulation, which is simpler in terms of computation complexity but provides only an approximation of the partition chart of the state space.

4.1. Tracing back from the equilibria

Taking advantage of the estimate of Ω from the Lyapunov function, we can perform a ‘trace-back’ for all the points in Ω . Based on our finding that $\mathbf{B}(\mathbf{x}^*)$ is strictly invariant, any point outside Ω that can dynamically evolve to a point inside Ω must also belong to $\mathbf{B}(\mathbf{x}^*)$. Thus, we can determine the inverse mapping of (7) and start the trace-back from Ω . This can be easily accomplished if the inverse function of (7) exists,

$$\mathbf{x}^{(n)} = \mathbf{y}^{(-1)}(\mathbf{x}^{(n+1)}). \tag{28}$$

For dynamical traffic systems whose recurrence functions are not invertible, we need to identify multiple points that can evolve to the same point in Ω . This procedure is repeated to gradually enlarge the estimate and is stopped when the enlargement is no longer deemed significant. In this approach, we expand the estimate Ω from the grid-search method, although there is still no guarantee that the entire region of $\mathbf{B}(\mathbf{x}^*)$ will be mapped out.

On the other hand, if we know the equilibrium solutions and their stability, based on Proposition 4, we can plot the exact boundaries of their attraction domains. For an asymptotically stable equilibrium \mathbf{x}^* , we know that $\partial\mathbf{B}(\mathbf{x}^*)$ is formed by trajectories to unstable equilibria. By first identifying the unstable equilibrium points and then tracing back the trajectories from these points, we can determine the boundaries of $\mathbf{B}(\mathbf{x}^*)$. And the exact region of $\mathbf{B}(\mathbf{x}^*)$ is simultaneously defined. This procedure of tracing $\partial\mathbf{B}(\mathbf{x}^*)$ can be done either numerically or analytically. In this first study, we illustrate the numerical method through some examples. We hope to provide the analytical derivation of determining $\partial\mathbf{B}(\mathbf{x}^*)$ in a follow-up study. Here we use one simple example with SUE to illustrate this concept. We note, however, that the method here is applicable to general networks with general travel cost functions.

Example 2c. Consider the dynamical system in Example 2. To estimate the attraction domains, a grid-search can be performed and the results are illustrated in Fig. 7a, where a rectangle estimate is established for both \mathbf{g}_i^* and \mathbf{g}_{III}^* . In this study, we can identify the exact range of the attraction domains by applying the results from Proposition 4: the boundaries of the attraction domains of stable equilibria are formed by the trajectories to the unstable equilibrium, i.e., \mathbf{g}_{II}^* in this example. Therefore, $\mathbf{B}(\mathbf{g}_i^*)$ and $\mathbf{B}(\mathbf{g}_{III}^*)$ are bounded and separated by the trajectories to \mathbf{g}_{II}^* . We can then trace-back from \mathbf{g}_{II}^* and collect all the points that converge to \mathbf{g}_{II}^* . Since the recurrence function is continuous, all points attracted to \mathbf{g}_{II}^* form a continuous curve. This curve, as shown by the dotted line in Fig. 3, functions as the boundary separating $\mathbf{B}(\mathbf{g}_i^*)$ and $\mathbf{B}(\mathbf{g}_{III}^*)$. The full range for $\mathbf{B}(\mathbf{g}_i^*)$ and $\mathbf{B}(\mathbf{g}_{III}^*)$ can therefore be identified as in Fig. 7b.

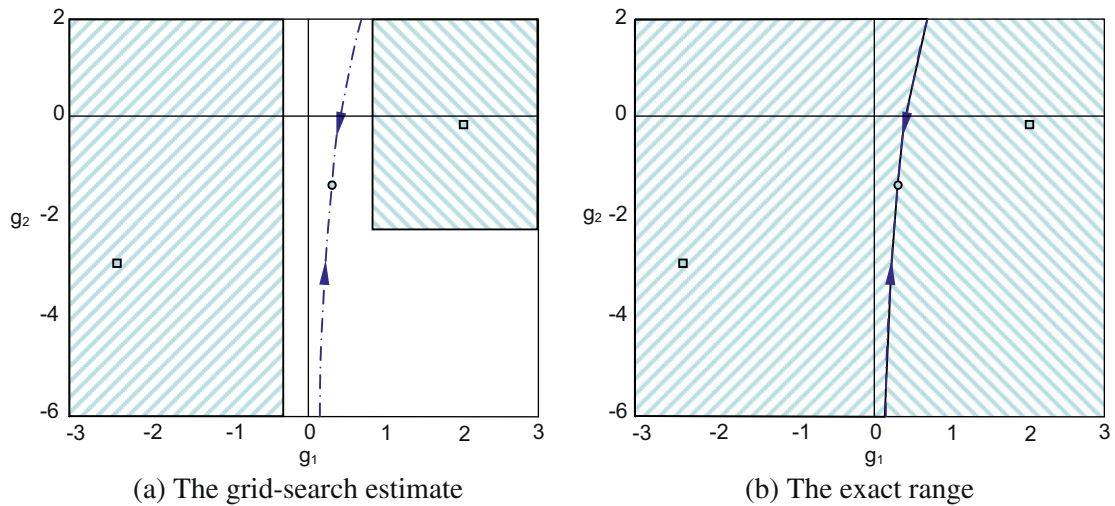


Fig. 7. Estimating the attraction domains.

4.2. Direction-based search by sampling

The method of tracing back from unstable equilibria can give us the exact boundaries of the attraction domains. We, however, note that unstable equilibria are sometimes difficult to precisely identify due to their instability or transient nature. Also, in practice we are not particularly interested in unstable equilibrium solutions; we are more interested in the stable equilibrium points and an adequate estimate of their attraction domains, such as an approximate partition chart of the state space. To achieve this, we can follow a sampling and simulation method to generate a rough picture of the phase portrait, and then perform a direction-based search to refine the estimated attraction domain. We demonstrate the procedure by an example.

Example 2d. Consider the dynamical system in Example 2. We now perform the direction-based search method to estimate the attraction domains. To do so, we first select a set of initial sampling points, here given by the integer points in the following set:

$$\{-3 < g_1 < 3; -6 < g_2 < 2\}.$$

There are totally 35 points, as shown in Fig. 8. We then trace their trajectories over time; for illustration we plot their states on day 3 (Fig. 9), day 10 (Fig. 10), day 30 (Fig. 11), and day 100 (Fig. 12). It is evident that the 21 initial points on the three columns on the left hand side converge to g_1^* while the 14 initial points on the right converge to g_{III}^* . We can then draw the attraction domain estimates as in Fig. 13. The estimation is made by connecting the points that are furthest from the equilibrium and these line segments form the boundary of the estimate. Estimate for $\mathbf{B}(g_1^*)$ is $\mathbf{E}(g_1^*) = \{g_1 \leq 0; -6 < g_2 < 2\}$ and estimate for $\mathbf{B}(g_{III}^*)$ is $\mathbf{E}(g_{III}^*) = \{g_1 \geq 1; -6 < g_2 < 2\}$. The uncharted range, $\{0 < g_1 < 1; -6 < g_2 < 2\}$, can be gradually reduced by selecting more points in the range and performing a second round of simulation.

We can expand the attraction domain estimates by reducing the uncharted range, i.e. the rectangle $\{0 < g_1 < 1; -6 < g_2 < 2\}$. In this round, similar to above, we choose as our samples the 28 points that are evenly distributed in the above range (Fig. 14). If we can trace their evolutions over time to the two stable equilibrium points, we will find out that the points represented by a cross evolve towards g_1^* while the points represented by a square evolve towards g_{III}^* . The resulting expansion of the attraction domain estimates is shown in Fig. 15.

We note here that this method also automatically obtains the stable equilibrium solutions. That is, even if we do not know the equilibrium solutions before performing the direction-based search, we can obtain the stable equilibrium solutions after a few steps of simulation. The selection of sample points above brings to mind the Monte Carlo method; the difference is that the sample points here are not randomly generated but evenly distributed. However, random samples can also be used to produce an estimate of the attraction domain, only that the shape of the estimate does not appear so regular as in Figs. 13 and 15.

If the procedure happens to step on the trajectory to the unstable equilibrium, then the problem is solved, as such a trajectory will reveal the boundary between the attraction domains. Otherwise, as shown in Fig. 15, the estimates can always be refined by another round of simulation for sample points in the uncharted range, with the estimated attraction domains expanded accordingly. As the direction-based search continues, the uncharted range will shrink and converge to a curve identical to the dotted line in Fig. 3, providing the boundary of the two attraction domains.

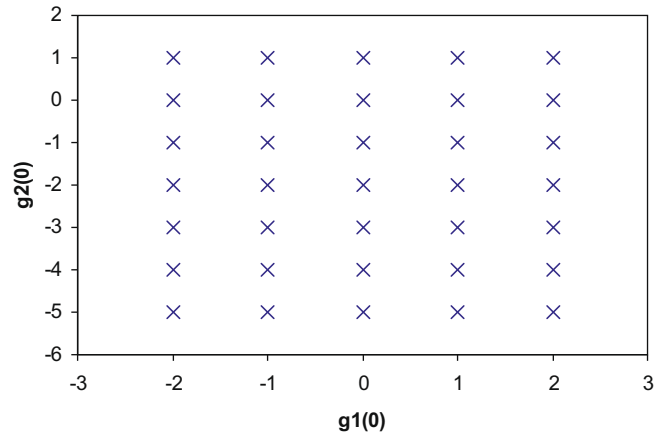


Fig. 8. Simulation points on day 0.

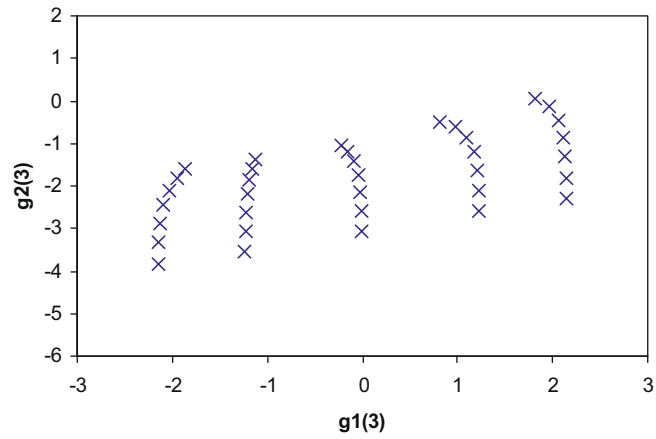


Fig. 9. Simulation points on day 3.

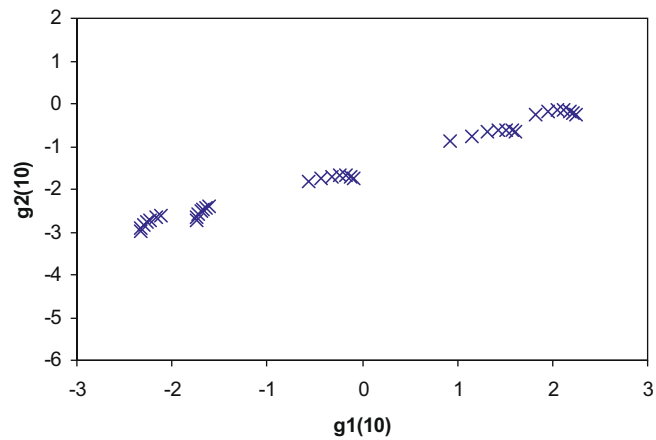


Fig. 10. Simulation points on day 10.

Here we have assumed that the attraction domain is connected and convex, i.e.

$$\lim_{n \rightarrow \infty} y^{(n)}(\mathbf{a}) = \lim_{n \rightarrow \infty} y^{(n)}(\mathbf{b}) = \mathbf{x}^* \Rightarrow \lim_{n \rightarrow \infty} y^{(n)}(\lambda \mathbf{a} + (1 - \lambda) \mathbf{b}) = \mathbf{x}^*, \quad \forall \lambda \in [0, 1]. \tag{29}$$

A convex set is one such that for any two points in the set, the line segment between the two points lies wholly in the set. If we know a number of points in the attraction domain, the line segment connecting any two of the points also belongs to the

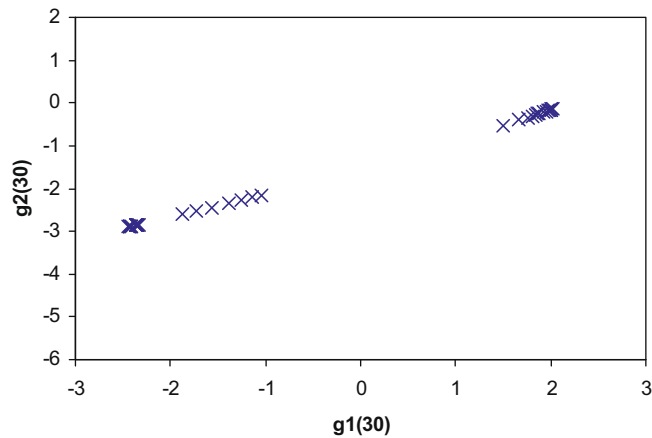


Fig. 11. Simulation points on day 30.

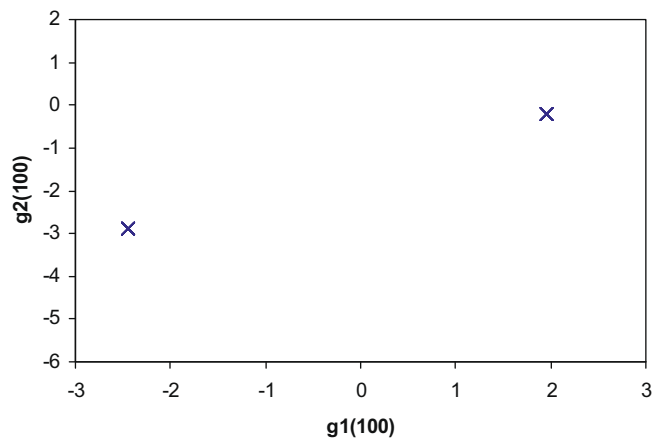


Fig. 12. Simulation points on day 100.

attraction domain. The estimate is then given as the biggest polytope formed by connecting the converging points as vertices, as shown by the two-dimensional case in the above example. For cases where the attraction domain is not convex, there would be some error included in the above attraction domain estimates, i.e. some non-convergent points may have been possibly enclosed in the polytope. This type of error can be reduced by gradually improving the precision of simulation with denser sample points.

5. Concluding remarks

In this paper, we studied some basic properties of the day-to-day traffic dynamics in pursuit of user equilibrium. We established the equivalency between static user equilibrium in traffic assignment and fixed point in dynamical traffic system. We investigated the stability issues of equilibrium and provided tools for determining and estimating the equilibrium's attraction domain. The equilibrium is only attainable when the dynamical evolution is started inside the attraction domain.

The dynamical system model of day-to-day traffic variations has not received much attention due to the general belief that equilibrium will prevail in the long run. Even if we accept this belief, it remains uncertain which equilibrium will prevail in situations with multiple equilibria. Only the dynamical system approach addresses the problem of how equilibrium is, or can be, achieved. Many studies on day-to-day traffic dynamics have chosen a continuous time dynamical system because of the desirable mathematical properties associated with continuous time systems. The real-life system of repeated daily trips is indisputably discrete. A future study topic is therefore to address the trade off between the two approaches and to possibly combine them for an improved result.

Travelers' route choice behavior is certainly more complex than the linear model in Section 2. Many of the simplifications adopted here can be relaxed in future studies. With the rapid development of ITS (intelligent transportation systems) technologies in recent years, daily traffic data become easier to obtain. One direction to go is to use the real traffic data to validate

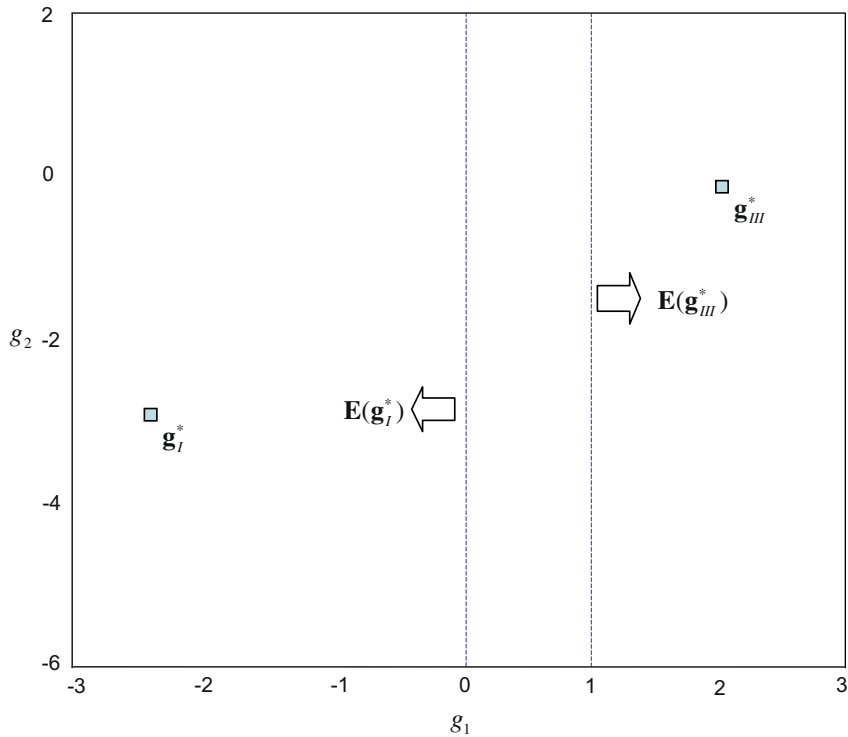


Fig. 13. Attraction domain estimates by direction-based search.

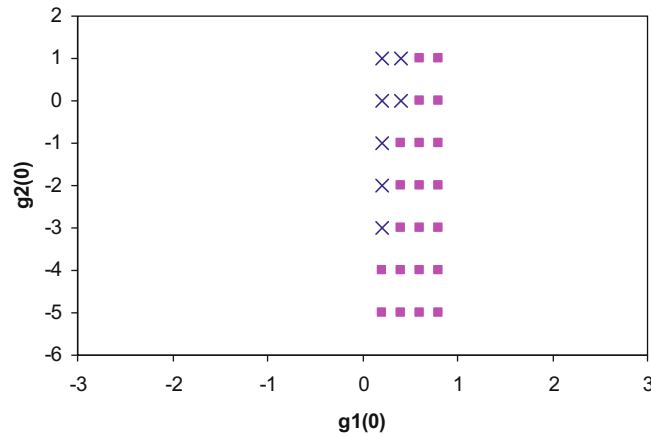


Fig. 14. Round-2 simulation points and their eventual convergence: $\times - g_I^*$, $\blacksquare - g_{III}^*$.

or calibrate the behavior model of day-to-day route choice. Because the network conditions such as weather vary from day to day, a difficulty of this study is to account for or remove these “noises” in the observed data. Another issue is the amount and type of data required, as collecting flow data on all links is too expensive and usually unnecessary. Besides, the effect of information provision on travelers’ route choice behavior is another topic of practical importance.

If the variation in travelers’ day-to-day trip-making includes not only route choice but also departure time choice, then one more dimension (i.e. departure time) can be added to the dynamical system. For the SUE system, an individual traveler’s knowledge of the network as represented by their perceived cost for a given day is then not a single value but a set containing the perceived costs for each departure time choice. Each day a traveler chooses the route and departure time that they perceive as the most beneficial. Equilibrium is formed when the perceived costs and the network flow are stationary from day to day.

Finally, we recognize that this paper is explorative in nature and is intended to scope out the problem and reveal some of its important properties. Looking ahead, we will continue the investigation on the properties of the attraction domain,

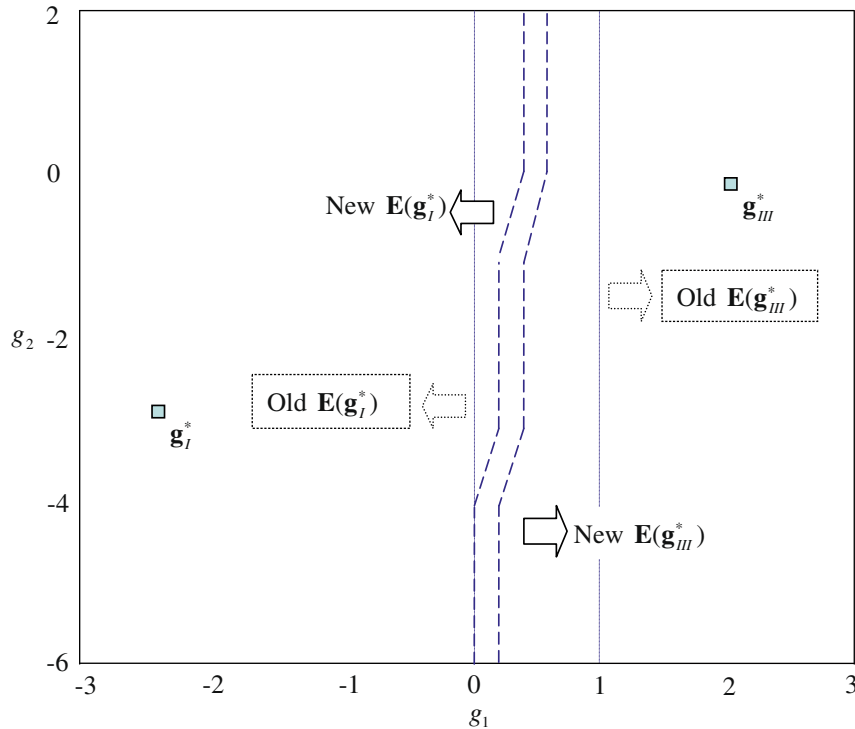


Fig. 15. Expanding the attraction domain estimates.

focusing on the theoretical foundation concerning its boundary and analytical identification. This will then be applied to a more comprehensive behavioral model, especially on general networks. We will also look into the implications for traffic management over time, such as guiding the traffic evolution to the desired direction.

Acknowledgement

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Appendix A. Proof of the equivalence between user equilibrium and fixed point in the DUE model

We start by showing that a fixed point as in (15) must also be a user equilibrium as in (3). Consider the fixed point \mathbf{f}^* . If route costs on the same OD pair are all equal, i.e. $c_k(\mathbf{f}^*) = c_j(\mathbf{f}^*)$, $\forall k, j \in \mathbf{R}_i, \forall i$, then (3) is obviously true; therefore \mathbf{f}^* is a user equilibrium. Otherwise route costs for some OD pair i are not equal for all the routes in \mathbf{R}_i . We will show then that only route(s) with the lowest cost can carry positive route flow, and all other routes have zero flow. To do so we first rank the OD routes from the highest cost to the lowest cost. We then select the route with the highest cost (or one out of the multiple routes with the highest cost), say, route j . From condition (17) we have $q_{ji}^* = 0$ for all route $l \in \mathbf{R}_i, l \neq j$. On the other hand, there is at least one route, say, route k , such that $c_k(\mathbf{f}^*) < c_j(\mathbf{f}^*)$. From condition (18) we have $q_{kj}^* > 0$; together with (13) this implies $q_{jj}^* < 1$. Eq. (15) gives

$$f_j^* = \sum_{l \in \mathbf{R}_i} q_{jl}^* w_{il}^* f_l^* + (1 - w_{jj}^*) f_j^*. \tag{30}$$

Equivalently,

$$\sum_{l \in \mathbf{R}_i, l \neq j} q_{jl}^* w_{il}^* f_l^* = (1 - q_{jj}^*) w_{jj}^* f_j^*. \tag{31}$$

The left hand side of (31) is equal to zero; for the right hand side, $1 - q_{jj}^* > 0, w_{jj}^* > 0$, therefore f_j^* must be zero. The same arguments hold for any other routes with the highest cost, i.e. there is no traffic flow on routes with the highest cost. For routes with the second highest cost, the same arguments above hold except that $q_{ji}^* = 0$ is not true for those route l 's with the highest cost; however, as we have just shown, those routes bears zero flow. So again we have the left hand side of

(31) equal to zero, and thus these routes with the second highest cost also have zero flow. Repeat this analysis and we reach the conclusion that only routes with the lowest route can carry positive flow. Subsequently the conditions for user equilibrium as in (3) are established.

Now we proceed to show that a user equilibrium as in (3) must also be a fixed point as in (15). Consider the user equilibrium $\mathbf{f}^{(n-1)}$. If $f_r^{(n-1)} > 0$ for route $r \in \mathbf{R}_i$, we have $c_r(\mathbf{f}^{(n-1)}) \leq c_s(\mathbf{f}^{(n-1)})$, $\forall s \in \mathbf{R}_i, s \neq r$. Together with (17), this gives $q_{sr}^{(n)} = 0$, $\forall s \in \mathbf{R}_i, s \neq r$; further with (13), this yields $q_{rr}^{(n)} = 1$. Now consider route $s \in \mathbf{R}_i, s \neq r$: if $c_s(\mathbf{f}^{(n-1)}) = c_r(\mathbf{f}^{(n-1)})$ we have $q_{rs}^{(n)} = 0$; if $c_s(\mathbf{f}^{(n-1)}) > c_r(\mathbf{f}^{(n-1)})$ we have $f_s^{(n-1)} = 0$ (because $f_s^{(n-1)} > 0$ would imply $c_s(\mathbf{f}^{(n-1)}) \leq c_r(\mathbf{f}^{(n-1)})$). Therefore we can derive

$$\begin{aligned} f_r^{(n)} &= \sum_{s \in \mathbf{R}_i} q_{rs}^{(n)} w_{ss}^{(n)} f_s^{(n-1)} + (1 - w_{rr}^{(n)}) f_r^{(n-1)} = \sum_{s \in \mathbf{R}_i, s \neq r} q_{rs}^{(n)} w_{ss}^{(n)} f_s^{(n-1)} + q_{rr}^{(n)} w_{rr}^{(n)} f_r^{(n-1)} - w_{rr}^{(n)} f_r^{(n-1)} + f_r^{(n-1)} \\ &= \sum_{s \in \mathbf{R}_i, s \neq r, c_s(\mathbf{f}^{(n-1)}) = c_r(\mathbf{f}^{(n-1)})} q_{rs}^{(n)} w_{ss}^{(n)} f_s^{(n-1)} + \sum_{s \in \mathbf{R}_i, s \neq r, c_s(\mathbf{f}^{(n-1)}) > c_r(\mathbf{f}^{(n-1)})} q_{rs}^{(n)} w_{ss}^{(n)} f_s^{(n-1)} + f_r^{(n-1)} = f_r^{(n-1)}. \end{aligned} \quad (32)$$

For route $r \in \mathbf{R}_i$ where $f_r^{(n-1)} = 0$, consider route $s \in \mathbf{R}_i, s \neq r$: if $f_s^{(n-1)} > 0$ then $c_s(\mathbf{f}^{(n-1)}) \leq c_r(\mathbf{f}^{(n-1)})$, implying $q_{rs}^{(n)} = 0$. We can then derive

$$\begin{aligned} f_r^{(n)} &= \sum_{s \in \mathbf{R}_i} q_{rs}^{(n)} w_{ss}^{(n)} f_s^{(n-1)} + (1 - w_{rr}^{(n)}) f_r^{(n-1)} \\ &= \sum_{s \in \mathbf{R}_i, s \neq r, f_s^{(n-1)} = 0} q_{rs}^{(n)} w_{ss}^{(n)} f_s^{(n-1)} + \sum_{s \in \mathbf{R}_i, s \neq r, f_s^{(n-1)} > 0} q_{rs}^{(n)} w_{ss}^{(n)} f_s^{(n-1)} + q_{rr}^{(n)} w_{rr}^{(n)} f_r^{(n-1)} + (1 - w_{rr}^{(n)}) f_r^{(n-1)} = 0. \end{aligned} \quad (33)$$

Therefore $\mathbf{f}^n = \mathbf{f}^{n-1}$ is also a fixed point of the dynamical system (12).

Appendix B. Proof of Proposition 1

The convergence of \mathbf{x}^* is equivalent to the statement below:

$$\forall \varepsilon > 0, \exists \delta_0 > 0, \exists N : \|\mathbf{x}^{(0)} - \mathbf{x}^*\| < \delta_0, n > N \Rightarrow \|\mathbf{x}^{(n)} - \mathbf{x}^*\| < \varepsilon.$$

That is, because the limit of the evolution is \mathbf{x}^* , the evolution will come arbitrarily close to \mathbf{x}^* after a sufficient large number of iterations. On the other hand, because $y(\cdot)$ is continuous, $y(y(\cdot))$, $y(y(y(\cdot)))$, \dots , $y^{(N)}(\cdot)$ are all continuous. Based on the Cauchy definition of continuity, for any $i = 1, 2, \dots, N$,

$$\forall \varepsilon > 0, \exists \delta_i > 0 : \|\mathbf{x}^{(0)} - \mathbf{x}^*\| < \delta_i \Rightarrow \|y^{(i)}(\mathbf{x}^{(0)}) - \mathbf{x}^*\| < \varepsilon.$$

Notice here that $y^{(i)}(\mathbf{x}^{(0)}) = \mathbf{x}^{(i)}$. Therefore if we set $\delta = \min \{\delta_0, \delta_1, \delta_2, \dots, \delta_N\}$, then

$$\forall \varepsilon > 0, \exists \delta > 0 : \|\mathbf{x}^{(0)} - \mathbf{x}^*\| < \delta \Rightarrow \|\mathbf{x}^{(n)} - \mathbf{x}^*\| < \varepsilon, \quad n = 1, 2, \dots, N, N + 1, \dots$$

This is exactly the definition for stability.

Appendix C. Proof of Proposition 2

To show that $\mathbf{B}(\mathbf{x}^*)$ is open, take any point $\mathbf{z} \in \mathbf{B}(\mathbf{x}^*)$, we need to show that there is a neighborhood of \mathbf{z} wherein every point converges to \mathbf{x}^* . We start by observing the following two points: Firstly, $\lim_{n \rightarrow +\infty} y^{(n)}(\mathbf{z}) = \mathbf{x}^*$ because $\mathbf{z} \in \mathbf{B}(\mathbf{x}^*)$; for any $\varepsilon_1 > 0$, we can then find a large enough N such that $\|y^{(N)}(\mathbf{z}) - \mathbf{x}^*\| < \varepsilon_1$. Secondly, the function y is continuous and so is $y^{(N)}$; by the Cauchy definition of continuity, for any $\varepsilon_2 > 0$, there exists a neighborhood of \mathbf{z} , say $\mathbf{H}(\mathbf{z}, \delta)$, such that $\mathbf{x} \in \mathbf{H}(\mathbf{z}, \delta) \Rightarrow \|y^{(N)}(\mathbf{x}) - y^{(N)}(\mathbf{z})\| < \varepsilon_2$. Then, for any point \mathbf{x} in $\mathbf{H}(\mathbf{z}, \delta)$, we have $\|y^{(N)}(\mathbf{x}) - \mathbf{x}^*\| \leq \|y^{(N)}(\mathbf{x}) - y^{(N)}(\mathbf{z})\| + \|y^{(N)}(\mathbf{z}) - \mathbf{x}^*\| < \varepsilon_1 + \varepsilon_2$. On the other hand, because \mathbf{x}^* is stable, there exists $\delta > 0$ such that $\|\mathbf{x}^{(0)} - \mathbf{x}^*\| < \delta \Rightarrow \mathbf{x}^{(0)} \in \mathbf{B}(\mathbf{x}^*)$. By setting $\varepsilon_1 + \varepsilon_2 \leq \delta$, we have $y^{(N)}(\mathbf{x}) \in \mathbf{B}(\mathbf{x}^*)$ for all $\mathbf{x} \in \mathbf{H}(\mathbf{z}, \delta)$. Therefore every point in the neighborhood $\mathbf{H}(\mathbf{z}, \delta)$ would converge to \mathbf{x}^* .

Appendix D. Proof of Proposition 3

The set $\partial \mathbf{B}(\mathbf{x}^*)$ is always non-empty unless $\mathbf{B}(\mathbf{x}^*) = \mathbf{D}$. Now we consider non-empty $\partial \mathbf{B}(\mathbf{x}^*)$; to prove that $\partial \mathbf{B}(\mathbf{x}^*)$ is invariant, we take any point $\mathbf{z} \in \partial \mathbf{B}(\mathbf{x}^*)$ and show that $y^{(n)}(\mathbf{z}) \in \partial \mathbf{B}(\mathbf{x}^*)$ for all $n = 1, 2, \dots$. First, we note that $\mathbf{z} \notin \mathbf{B}(\mathbf{x}^*)$ because $\mathbf{B}(\mathbf{x}^*)$ is open according to Proposition 2. Moreover, as $\mathbf{B}(\mathbf{x}^*)$ is strictly invariant, we have the result that $y^{(n)}(\mathbf{z}) \notin \mathbf{B}(\mathbf{x}^*)$ for all n . That is, the trajectories of any point started from $\partial \mathbf{B}(\mathbf{x}^*)$ shall never become part of $\mathbf{B}(\mathbf{x}^*)$. However, since \mathbf{z} is on the boundary of $\mathbf{B}(\mathbf{x}^*)$, any neighborhood of \mathbf{z} has a non-empty intersection with $\mathbf{B}(\mathbf{x}^*)$. As such, we can always find an infinite sequence of points in $\mathbf{B}(\mathbf{x}^*)$ that takes \mathbf{z} as its limit, i.e. a sequence of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_i, \dots \in \mathbf{B}(\mathbf{x}^*)$ such that $\lim \mathbf{x}_i = \mathbf{z}$. Given that y is a continuous mapping, according to the Heine definition of continuity, $\lim \mathbf{x}_i = \mathbf{z} \Rightarrow \lim y(\mathbf{x}_i) = y(\mathbf{z})$. By repeatedly applying this definition, we have that $\lim y^{(n)}(\mathbf{x}_i) = y^{(n)}(\mathbf{z})$ for all n . So $y^{(n)}(\mathbf{z})$ is the limit of the sequence $y^{(n)}(\mathbf{x}_1), y^{(n)}(\mathbf{x}_2), \dots, y^{(n)}(\mathbf{x}_i), \dots \in \mathbf{B}(\mathbf{x}^*)$. That is, $\lim_{i \rightarrow \infty} y^{(n)}(\mathbf{x}_i)$ approaches $y^{(n)}(\mathbf{z})$ but $y^{(n)}(\mathbf{z}) \notin \mathbf{B}(\mathbf{x}^*)$. Combining these two facts, we can conclude that $y^{(n)}(\mathbf{z})$ is on the boundary

of $\mathbf{B}(\mathbf{x}^*)$, i.e. $y^{(n)}(\mathbf{z}) \in \partial\mathbf{B}(\mathbf{x}^*)$ for all n . Therefore we have shown that $\partial\mathbf{B}(\mathbf{x}^*)$ is invariant and any trajectory starting in $\partial\mathbf{B}(\mathbf{x}^*)$ shall remain forever in $\partial\mathbf{B}(\mathbf{x}^*)$. Hence $\partial\mathbf{B}(\mathbf{x}^*)$ is formed by trajectories.

Appendix E. Proof of Proposition 4

We have shown that $\partial\mathbf{B}(\mathbf{x}^*)$ is formed by trajectories. In the following, we will show that none of these trajectories can be trajectories towards a stable fixed point and therefore they must be trajectories to unstable fixed points. Suppose that a certain trajectory on $\partial\mathbf{B}(\mathbf{x}^*)$ converges to another stable fixed point \mathbf{x}'' , then any point \mathbf{z} on this trajectory belongs to the open set $\mathbf{B}(\mathbf{x}'')$. Therefore, there must exist a neighborhood of \mathbf{z} that lies entirely inside $\mathbf{B}(\mathbf{x}'')$. On the other hand, any neighborhood of \mathbf{z} will overlap with $\mathbf{B}(\mathbf{x}^*)$. This contradicts the fact that the domains of attraction for two different fixed points are mutually exclusive and cannot overlap. Hence, no trajectory on $\partial\mathbf{B}(\mathbf{x}^*)$ converges to a stable fixed point.

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